Manifold Notes

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Abstract

This is the Imperial College London postgraduate module MATH70058 Manifolds, instructed by Dr. Yuhan Sun. The formal name for this class is “Manifolds”.

Contents

Course information

Office hours are in Huxley 6M22 on Friday at 16:00.

The materials covered in course will be:

1. the general definitions and structure of Manifolds,
2. functions on manifolds such as, derivatives and integrals, and
3. vector bundles.

1 Smooth Manifolds

Note 1.1. In the simplest terms, 'smooth manifolds' are spaces that locally look like some Euclidean space $\mathbb{R}^n$, and on which one can do calculus.

1.1 Topological Manifolds

Note 1.2. The simplest 'manifold' are the topological manifolds, which are topological spaces with certain properties that encode waht we mean when we say that they "locally look like" $\mathbb{R}^n$.

Definition 1.3. Let (X, τ) be a topological space. A subcollection $\mathcal{B}$ of τ is a **basis** for τ if every open set in X is a union of sets in $\mathcal{B}$.

Example 1.4. Consider the topological space $\mathbb{R}^n$ with the usual topology, then the open n-balls provide a basis of the topology.

Definition 1.5. Suppose M is a topological space. We say that M is a **topological manifold of dimension** n or a **topological** n**-manifold** if it has the following properties:

1. M is Hausdorff space, i.e. for every pair of distinct points $p, q \in M$ there exists disjoint open subsets $U, V \subseteq M$ such that $p \in U$ and $q \in V$.
2. M is second-countable, i.e. there exists a countable basis for the topology of M.
3. M is locally Euclidean of dimension n, i.e. each point of M has a neighbourhood that is homeomorphic to an open subset of $\mathbb{R}^n$.

Remark 1.6. The third property means that for each $p \in M$ we can find

- an open subset $U \subseteq M$ containing p,
- an open subset $\widehat{U} \subseteq \mathbb{R}^n$, and
- a homeomorphism $\varphi : U \to \widehat{U}$.

Note 1.7. We may sometimes write "Let M^n be a manifold" as a shorthand for "Let M be a topological manifold of dimension n".

Proposition 1.8
Some useful facts:

1. $\mathbb{R}^n$ and every metric space is a second countable Hausdorff space.
2. Subset of second countable Hausdorff spaces are second countable Hausdorff spaces.

Note 1.9. So any subspace of $\mathbb{R}^n$ is automatically Hausdorff and second countable.

Proof. Refer to [Tu10, Proposition A.19] and [Tu10, Proposition A.14] respectively. □

1.1.1 Coordinate charts

Definition 1.10. Let M be a topological n-manifold. A **coordinate chart** (or just **chart**) on M is a pair (U, φ), where U is an open subset of M and $\varphi : U \to \varphi(U) \subseteq \mathbb{R}^n$ is a homeomorphism.

The map φ is called a (**local**) **coordinate map**, and the component functions $(x_1, \ldots, x_n)$ of φ, defined by $\varphi(p) = (x_1(p), \ldots, x_n(p))$, are called **local coordinates** on U. Figure 1 is an illustration of a chart.

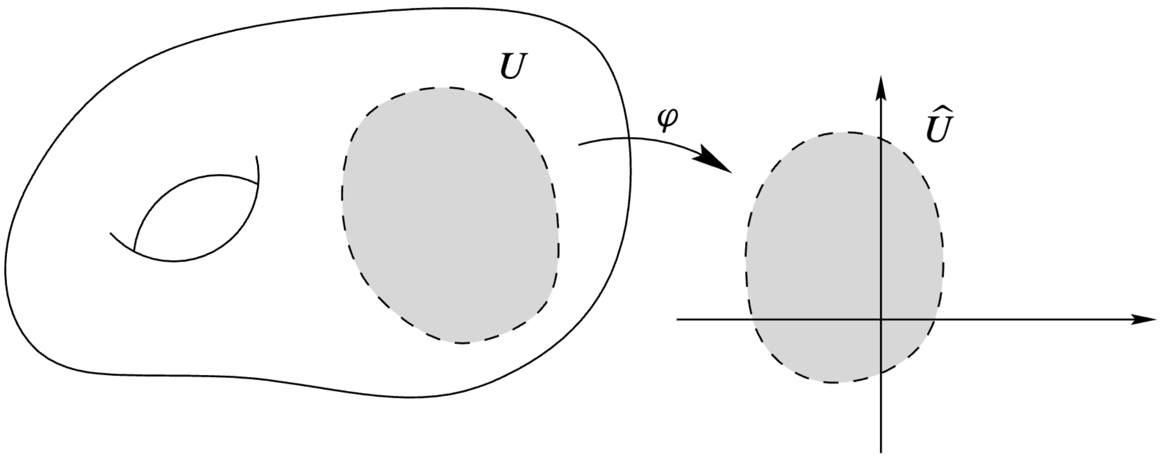

Figure 1: A chart [Lee12, Fig. 1.2].

1.1.2 Examples of topological manifolds

Example 1.11. The Euclidean space $\mathbb{R}^n$ is covered by a single chart $(\mathbb{R}^n, \mathbf{1}_{\mathbb{R}^n})$, where $\mathbf{1}_{\mathbb{R}^n} : \mathbb{R}^n \to \mathbb{R}^n$ is the identity map. Furthermore, every open subset of $\mathbb{R}^n$ is also a topological manifold, with chart $(U, \mathbf{1}_U)$.

Remark 1.12. Recall that the Hausdorff condition and second countability are "hereditary properties"; that is, they are inhereted by subspaces.

Example 1.13

A cusp, the graph $y = x^{2/3}$ in $\mathbb{R}^2$ is a topological manifold, by virtue of being a subspace of $\mathbb{R}^2$. It it is locally Euclidean, because it is homeomorphic to $\mathbb{R}$ via $(x, x^{2/3}) \mapsto x$.

Example 1.14 (Graph of smooth functions [Lee12, Example 1.3])**.** Let $U \subseteq \mathbb{R}^n$ be an open subset, and let $f : U \to \mathbb{R}^k$ be a continuous function. The *graph of* f is the subset of $\mathbb{R}^n \times \mathbb{R}^k$ defined by

$$\Gamma(f) = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^k : x \in U \text{ and } y = f(x)\},$$

with the subspace topology. Let $\pi_1 : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$ denote the projection onto the first factor, and let $\varphi : \Gamma(f) \to U$ be the restriction of π_1 to $\Gamma(f)$:

$$\varphi(x, y) = x, \quad (x, y) \in \Gamma(f).$$

Because φ is the restriction of a continuous map, it is continuous; and it is a homeomorphism because it has a continuous inverse given by $\varphi^{-1}(x) = (x, f(x))$. Thus $\Gamma(f)$ is a topological manifold of dimension n. In fact, $\Gamma(f)$ is homeomorphic to U itself, and $(\Gamma(f), \varphi)$ is a global coordinate chart, called *graph coordinates*. The same observation applies to any subset of $\mathbb{R}^{n+k}$ defined by setting any k of the coordinates (not necessarily the last k) equal to some continuous function of the other n, which are restricted to lie in an open subset of $\mathbb{R}^n$.

Example 1.15 (Spheres [Lee12, Example 1.4])**.** For each integer $n \geq 0$, the unit n-sphere $\mathbb{S}^n$ is Hausdorff and second-countable because it is a topological subspace of $\mathbb{R}^{n+1}$. To show that it is locally Euclidean, for each index $i = 1, \ldots, n+1$, let U_i^+ denote the subset of $\mathbb{R}^{n+1}$ where the i-th coordinate is positive:

$$U_i^+ = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_i > 0\}.$$

Similarly, U_i^- is the set where $x_i < 0$.

Let $f : B^n \to \mathbb{R}$ (where B^n is the open unit ball) be the continuous function

$$f(u) = \sqrt{1 - |u|^2}.$$

Then for each $i = 1, \ldots, n+1$, it is easy to check that $U_i^+ \cap \mathbb{S}^n$ is the graph of the function

$$x_i = f(x_1, \ldots, \widehat{x_i}, \ldots, x_{n+1}),$$

where the hat indicates that x_i is omitted. Similarly, $U_i^- \cap \mathbb{S}^n$ is the graph of

$$x_i = -f(x_1, \ldots, \widehat{x_i}, \ldots, x_{n+1}).$$

Thus, each subset $U_i^\pm \cap \mathbb{S}^n$ is locally Euclidean of dimension n, and the maps

$$\varphi_i^\pm : U_i^\pm \cap \mathbb{S}^n \to B^n$$

given by

$$\varphi_i^\pm(x_1, \ldots, x_{n+1}) = (x_1, \ldots, \widehat{x_i}, \ldots, x_{n+1})$$

are graph coordinates for $\mathbb{S}^n$. Since each point of $\mathbb{S}^n$ is in the domain of at least one of these $2n + 2$ charts, $\mathbb{S}^n$ is a topological n-manifold.

Example 1.16 (Product Manifolds [Lee12, Example 1.8]). Suppose $M_1, \dots, M_k$ are topological manifolds of dimensions $n_1, \dots, n_k$, respectively. The product space $M_1 \times \dots \times M_k$ is shown to be a topological manifold of dimension $n_1 + \dots + n_k$ as follows. It is Hausdorff and second-countable, so only the locally Euclidean property needs to be checked. Given any point $(p_1, \dots, p_k) \in M_1 \times \dots \times M_k$, we can choose a coordinate chart (U_i, φ_i) for each M_i with $p_i \in U_i$. The product map

$$\varphi_1 \times \dots \times \varphi_k : U_1 \times \dots \times U_k \to \mathbb{R}^{n_1+\dots+n_k}$$

is a homeomorphism onto its image, which is a product open subset of $\mathbb{R}^{n_1+\dots+n_k}$. Thus, $M_1 \times \dots \times M_k$ is a topological manifold of dimension $n_1 + \dots + n_k$, with charts of the form $(U_1 \times \dots \times U_k, \varphi_1 \times \dots \times \varphi_k)$.

Example 1.17 (Tori [Lee12, Example 1.9]). For a positive integer n, the n-torus (plural: tori) is the product space

$$\mathbb{T}^n = \mathbb{S}^1 \times \dots \times \mathbb{S}^1.$$

By the discussion above, it is a topological n-manifold. (The 2-torus is usually called simply *the torus*.)

Example 1.18 (Non-examples)

We provide examples in which one of the three properties to be a topological manifold is not satisfied.

- [NOT Hausdorff]. We consider a "line with two origins". To construct this space, we begin with the set $S = \{(x, y) \in \mathbb{R}^2 : y = \pm 1\} \subseteq \mathbb{R}^2$, which consists of two horizontal lines, one at $y = 1$ and the other at $y = -1$, both with the subspace topology from $\mathbb{R}^2$.

  We introduce an equivalence relation $\sim$ on S, defined as $(x, y) \sim (x', y')$ if and only if $x = x' \neq 0$. This relation identifies points on the two lines that have the same x-coordinate, except at the origin, where the points $(0, 1)$ and $(0, -1)$ remain distinct. The "line with two origins" is the quotient space $S/\sim$, equipped with the quotient topology.

  In this space, the points corresponding to $(0, 1)$ and $(0, -1)$ cannot be separated by disjoint open sets.

- [NOT second countable]. Consider the disjoint union of uncountably many copies of $\mathbb{R}$.

- [NOT locally Euclidean]. Consider the set $S = \{(x, y) \in \mathbb{R}^2 : xy = 0\}$, which is the union of the coordinate axes in $\mathbb{R}^2$, with the subspace topology. The point $(0, 0)$ fails to be locally Euclidean because any neighbourhood of $(0, 0)$ in S is not homeomorphic to an open subset of $\mathbb{R}$. If we remove $(0, 0)$, the neighbourhood splits into four disconnected branches (the positive and negative parts of the axes), unlike any connected open set in $\mathbb{R}$. Therefore, S is not locally Euclidean at the origin.

1.2 Smooth structures

Note 1.19. The definition of manifolds that we gave in the preceding section is sufficient for studying topological properties of manifolds. But now we want to do calculus on manifolds.

Definition 1.20. If $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ are open subsets, a function $F : U \to V$ is said to be **smooth** (or $F \in C^\infty$) if each of its component functions that has continuous partial derivatives of all orders.

Definition 1.21. Let M be a topological n-manifold. If (U, φ) and (V, ψ) are two charts such that $U \cap V \neq \varnothing$, the composite map $\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$ is called the **transition map from φ to ψ**. Figure 2 is an illustration of a transition map.

Remark 1.22. In the lectures the transition map is given by $\varphi \circ \psi^{-1} : \psi(U \cap V) \to \varphi(U \cap V)$. The two formulations are completely equivalent, but we choose the former to stay in line with [Lee12].

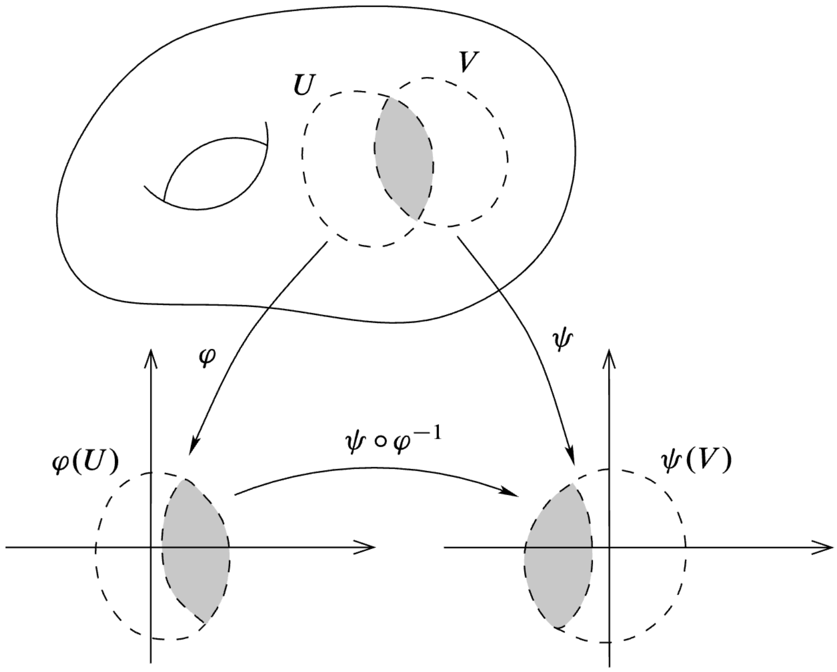

Figure 2: A transition map [Lee12, Fig. 1.6]

Corollary 1.23. The transition map is a homeomorphism.

Proof. Composition of homeomorphisms is a homeomorphism. $\square$

Definition 1.24. Two charts (U, φ) and (V, ψ) are said to be **smoothly compatible** if either $U \cap V = \varnothing$ or the transition map $\psi \circ \varphi^{-1}$ is a diffeomorphism.

Remark 1.25. Since $\varphi(U \cap V)$ and $\psi(U \cap V)$ are open subsets of $\mathbb{R}^n$, smoothness of this map is to be interpreted in the ordinary sense of having continuous partial derivatives of all orders.

Definition 1.26. A **smooth atlas**, $\mathcal{A} = \{(U_i, \varphi_i) : i \in I\}$, for a manifold M^n is a collection of charts such that:

1. $\bigcup_{i \in I} U_i = M$, and
2. any two charts (U_i, φ_i) and (U_j, φ_j) are smoothly compatible.

Note 1.27. An atlas is a collection of charts whose domains cover M; this collection is smooth if any pair of charts of the atlas is smoothly compatible.

Example 1.28. Some examples of smooth atlases:

- $\mathcal{A} = \{(\mathbb{R}, \mathrm{id}_{\mathbb{R}})\}$ is an atlas for the manifold $\mathbb{R}$. We
- Let $U \subset \mathbb{R}^n$ be an open subset. Consider U as a topological manifold then the single chart (U, id_U) is smooth atlas for U.
- $\mathcal{A} = \left\{((a_i, b_i), \mathrm{id}_{(a_i, b_i)}) : i \in I, a_i, b_i \in \mathbb{R}\right\}$ is an atlas for $\mathbb{R}$. The charts in this case are all kinds of open intervals in $\mathbb{R}$.

Example 1.29 (Non-example of atlas)

Let $\mathbb{R}$ be a manifold. Consider two charts for $\mathbb{R}$: $(\mathbb{R}, \mathrm{id}_{\mathbb{R}})$ and $(\mathbb{R}, \psi(x) = x^3)$. These are NOT smoothly compatible because ψ^{-1} is not differentiable at zero hence, cannot be smooth.

Example 1.30 (Atlas on a circle)
The unit circle in the complex plane may be described by as the set of points $\{e^{it} : 0 \leq t \leq 2\pi\}$. Let U_1 and U_2 be two open subsets of S^1:

$$U_1 = \left\{e^{it} : -\pi < t < \pi\right\},$$
$$U_2 = \left\{e^{it} : 0 < t < 2\pi\right\}$$

and define $\phi_\alpha : U_\alpha \to \mathbb{R}$ for $\alpha = 1, 2$ by

$$\phi_1(e^{it}) = t \quad -\pi < t < \pi$$
$$\phi_2(e^{it}) = t \quad 0 < t < 2\pi.$$

Both ϕ_1 and ϕ_2 are branches of the complex log funcion $(1/i)\log(z)$ and are homeomorphism onto their respective images. Thus, (U_1, ϕ_1) and (U_2, ϕ_2) are charts on S^1. The intersection $U_1 \cap U_2$ consists of two connected components,

$$A = \left\{e^{it} : -\pi < t < 0\right\}, B = \{0 < t < \pi\},$$

with

$$\phi(U_1 \cap U_2)_1 = \phi_1(A \sqcup B) = \phi_1(A) \sqcup \phi_1(B)$$
$$\phi(U_1 \cap U_2)_2 = \phi_2(A \sqcup B) = \phi_2(A) \sqcup \phi_2(B).$$

Proposition 1.31
To show that an atlas is smooth, we need only verify that each transition map $\psi \circ \varphi^{-1}$ is smooth whenever (U, φ) and (V, ψ) are charts in $\mathcal{A}$; once we have proved this, it follows that $\psi \circ \varphi^{-1}$ is a diffeomorphism because its inverse

$$(\psi \circ \varphi^{-1})^{-1} = \varphi \circ \psi^{-1}$$

is one of the transition maps we have already shown to be smooth. Alternatively, given two particular charts (U, φ) and (V, ψ), it is often easiest to show that they are smoothly compatible by verifying that $\psi \circ \varphi^{-1}$ is smooth and injective with nonsingular Jacobian at each point, and appealing to Corollary A.1.

Example 1.32. We present the atlas of the previous examples.

- Graph of smooth function. There is a single chart given by $(\Gamma(f), \phi)$.
- Product manifold [Tu10, Proposition 5.18]. If $\{(U_i, \phi_i) : i \in I\}$ and $\{(V_j, \psi_j) : j \in J\}$ are smooth atlases for the manifolds M and N of dimension m and n, respectively, then the collection

$$\{U_i \times V_j, \phi_i \times \psi_j : U_i \times V_j \to \mathbb{R}^m \times \mathbb{R}^n\}$$

  of charts is a smooth atlas on $M \times N$.

Definition 1.33. Suppose $f : \mathbb{R}^n \to \mathbb{R}^m$ is a function such that each of its first-order partial derivatives exists on $\mathbb{R}^n$. Then the **Jacobian matrix** of f, denoted $J_f \in \mathbb{R}^{n \times m}$

is defined such that its $(i,j)^{\text{th}}$ entry is $\dfrac{\partial f_i}{\partial x_i}$. That is,

$$J_f = \begin{pmatrix} \dfrac{\partial f_1}{\partial x_1} & \cdots & \dfrac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial f_m}{\partial x_1} & \cdots & \dfrac{\partial f_m}{\partial x_n} \end{pmatrix}.$$

Lemma 1.34. Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be open subsets. If U is diffeomorphic to V then $n = m$.

Proof. By the chain rule, the Jacobian matrix of f is the inverse of the Jacobian matrix of f^{-1}. For the inverse to exists, these two matrices must be square, hence $n = m$. □

Remark 1.35. We use the above idea to disprove that a given space is locally Euclidean. The strategy is to assume that two spaces $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ are homeomorphic. Since the dimension is a topological invariant under homeomorphism, it must follow that $n = m$. We then identify a property preserved under homeomorphism — such as path-connectedness — that leads to a contradiction, thereby ruling out the possibility of local Euclidean structure.

Definition 1.36. A smooth atlas $\mathcal{A}$ on M is **maximal** (or **complete**) if any chart that is smoothly compatible with every chart in $\mathcal{A}$ is already in $\mathcal{A}$. We denote a maximal atlas by $m\mathcal{A}$.

Note 1.37. An atlas is maximal if it contains all the charts which are compatible with each other.

Example 1.38. Consider the atlas $\mathcal{A} = \{(\mathbb{R}, \mathrm{id}_{\mathbb{R}})\}$. This atlas is not maximal as it does not contain the chart $(0,1), \mathrm{id}_{(0,1)}$. A non-trivial example of a smoothly compatible chart is $(\mathbb{R}, e^x)$.

Definition 1.39. If M is a topological manifold, a **smooth structure on** M is a maximal smooth atlas.

Definition 1.40. A **smooth manifold** is a pair $(M, \mathcal{A})$, where M is a topological manifold and $\mathcal{A}$ is a smooth structure on M.

Example 1.41

Not every topological manifold admits a smooth structure. This is a non-trivial example provided by Michel Kervaire.

Proposition 1.42

Let M be a topological manifold.

1. Every smooth atlas $\mathcal{A}$ for M is contained in a unique maximal smooth atlas, called the **smooth structure determined by** $\mathcal{A}$.

2. Two smooth atlases for M determine the same smooth structure if and only if their union is a smooth atlas.

Note 1.43. Given two maximal atlases $m\mathcal{A}$ and $m\mathcal{B}$ we have that $m\mathcal{A} = m\mathcal{B} \iff \mathcal{A} \cup \mathcal{B}$ is a smooth atlas.

Proof. We prove each statement in turn.

1. Let $\mathcal{A}$ be a smooth atlas for M, and let $m\mathcal{A}$ be the collection of all charts on M that are smoothly compatible with every chart in $\mathcal{A}$. By definition $m\mathcal{A}$ already covers the whole space since $\mathcal{A}$ is already an atlas. We need to show $m\mathcal{A}$ is a smooth atlas i.e. that any two charts are smoothly compatible. Pick two charts $(U, \varphi), (V, \psi) \in m\mathcal{A}$, the goal is to show that $\varphi \circ \psi^{-1}$ is a diffeomorphism. If $U \cap V = \varnothing$ then we have nothing to prove as it is smoothly compatible. Suppose $U \cap V \neq \varnothing$ then we can pick a point $p \in U \cap V$. Since we know $\mathcal{A}$ is an atlas for M there exists some chart, say $(W, \theta) \in \mathcal{A}$ such that $p \in W$. We can write $\varphi \circ \psi^{-1} = (\varphi \circ \theta^{-1}) \circ (\theta \circ \psi^{-1})$, where each $(\square \circ \square)$ is a diffeomorphism by the hence, the composition is also a diffeomorphism. Figure 3 illustrates the proof.

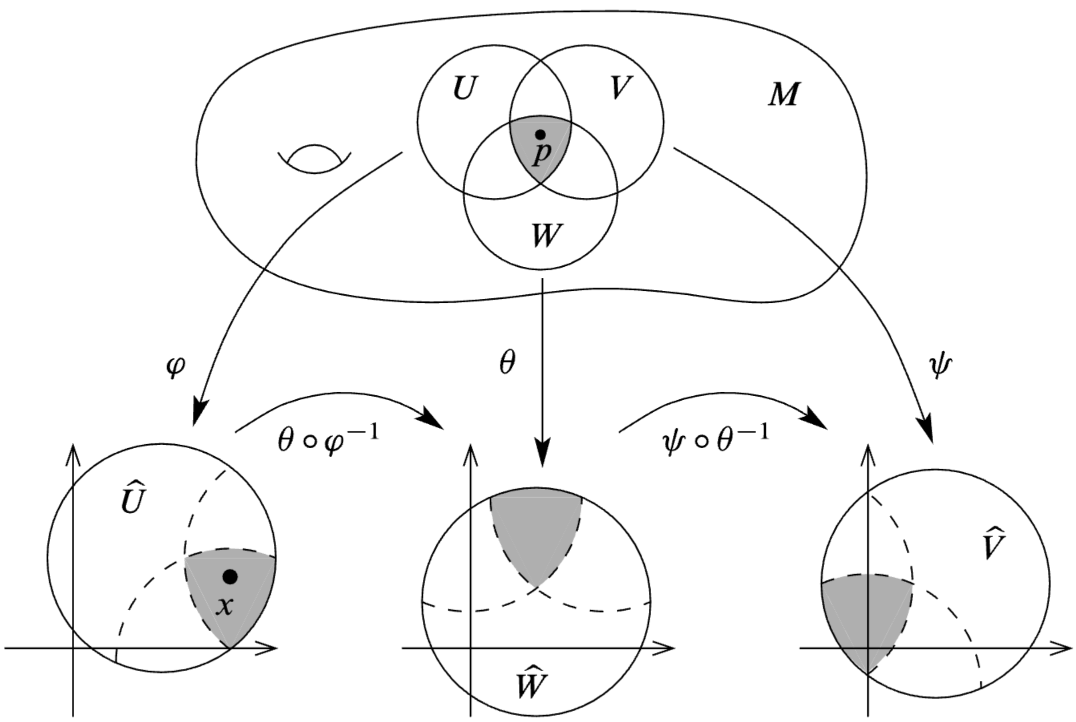

Figure 3: Proof of Proposition 1.25 [Lee12, Fig. 1.7]

The above proves the exists of a maximal smooth atlas containing $\mathcal{A}$. To prove the uniqueness we argue by contradiction. Suppose $\mathcal{B}$ is any other maximal smooth atlas containing $\mathcal{A}$, each of its charts is smoothly compatible with each chart in $\mathcal{A}$, so $\mathcal{B} \subseteq m\mathcal{A}$. By maximality of $\mathcal{B}$ we must have $\mathcal{B} = m\mathcal{A}$.

2. Let $\mathcal{A}$ and $\mathcal{B}$ be two smooth atlases of the manifold. We prove each direction in turn.

   - Proof of ($\Rightarrow$). Trivial by considering that $\mathcal{B} \subseteq m\mathcal{B} \subseteq m\mathcal{A}$.

- Proof of ($\Leftarrow$). Any chart in $\mathcal{B}$ is smoothly compatible with any chart of $\mathcal{A}$ then by definition $\mathcal{B} \subseteq m\mathcal{A}$. Similarly, $A \subseteq m\mathcal{B}$. We know $B \subseteq m\mathcal{B}$ and by using the uniqueness of the maximal atlas we have the desired equality.

$\square$

Example 1.44. Consider two smooth atlases of $\mathbb{R}$ given by $\mathcal{A} = \{(\mathbb{R}, \mathrm{id}_{\mathbb{R}})\}$ and $\mathcal{B} = \{(\mathbb{R}, x^3)\}$. We have that $\mathcal{A} \cup \mathcal{B}$ is not a smooth atlas as the two charts are not smoothly compatible hence, $\mathcal{A}$ and $\mathcal{B}$ determine distinct smooth structures of $\mathbb{R}$.

1.3 Examples of smooth manifolds

Example 1.45. The n-dimensional Euclidean space $\mathbb{R}^n$, equipped with a single chart $(\mathbb{R}^n, \varphi(x) = x)$ is a smooth manifold. We say this smooth structure is the standard one on $\mathbb{R}^n$.

Example 1.46. Any n-dimensional vector space E over $\mathbb{R}$ can be equipped with a smooth structure by identifying E with $\mathbb{R}^n$.

Example 1.47 (Spaces of matrices)

Some examples regarding matrices.

- Let $\mathrm{M}(m \times n, \mathbb{R})$ denote the set of $m \times n$ matrices with real entries. Because it is a real vector space of dimension mn under matrix addition and scalar multiplication, $\mathrm{M}(m \times n, \mathbb{R})$ is a smooth mn-dimensional manifold.

- The **general linear group** $\mathrm{GL}(n, \mathbb{R})$ is the set of invertible $n \times n$ matrices with real entries. It is a smooth n^2-dimensional manifold because it is an open subset of the n^2-dimensional vector space $\mathrm{M}(n, \mathbb{R})$, namely the set where the (continuous) determinant function is nonzero.

- (**Matrices of Full Rank**) The previous example has a natural generalisation to rectangular matrices of full rank. Suppose $m < n$, and let $\mathrm{M}_m(m \times n, \mathbb{R})$ denote the subset of $\mathrm{M}(m \times n, \mathbb{R})$ consisting of matrices of rank m. If A is an arbitrary such matrix, the fact that $\operatorname{rank} A = m$ means that A has some nonsingular $m \times m$ submatrix. By continuity of the determinant function, this same submatrix has nonzero determinant on a neighborhood of A in $\mathrm{M}(m \times n, \mathbb{R})$, which implies that A has a neighborhood contained in $\mathrm{M}_m(m \times n, \mathbb{R})$. Thus, $\mathrm{M}_m(m \times n, \mathbb{R})$ is an open subset of $\mathrm{M}(m \times n, \mathbb{R})$, and therefore is itself a smooth mn-dimensional manifold. A similar argument shows that $\mathrm{M}_n(m \times n, \mathbb{R})$ is a smooth mn-manifold when $n < m$.

Example 1.48 (Graphs of smooth functions)**.** If $U \subseteq \mathbb{R}^n$ is an open subset and $f : U \to \mathbb{R}^k$ is a smooth function, we have already observed above (Examle 1.14) that the graph of f is a topological n-manifold in the subspace topology. Since $\Gamma(f)$ is covered by the single graph coordinate chart $\varphi : \Gamma(f) \to U$ (the restriction of π_1), we can put a canonical smooth structure on $\Gamma(f)$ by declaring the graph coordinate chart $(\Gamma(f), \varphi)$ to be a smooth chart.

Example 1.49. Let M be a smooth manifold and $U \subseteq M$ be an open subset. Then U itself is a smooth manifold, with the smooth structure induced from M,

Example 1.50

Example 1.31 (Spheres). We showed in Example 1.15 that the n-sphere $\mathbb{S}^n \subseteq \mathbb{R}^{n+1}$ is a topological n-manifold. We put a smooth structure on $\mathbb{S}^n$ as follows. For each $i = 1, \ldots, n+1$, let $(U_i^\pm, \varphi_i^\pm)$ denote the graph coordinate charts we constructed in Example 1.15. For any distinct indices i and j, the transition map $\varphi_i^\pm \circ (\varphi_j^\pm)^{-1}$ is easily computed. In the case $i < j$, we get

$$\varphi_i^\pm \circ (\varphi_j^\pm)^{-1}(u_1, \ldots, u_n) = (u_1, \ldots, \widehat{u_i}, \ldots, \pm\sqrt{1 - |u|^2}, \ldots, u_n)$$

(with the square root in the j-th position), and a similar formula holds when $i > j$. When $i = j$, an even simpler computation gives

$$\varphi_i^+ \circ (\varphi_i^-)^{-1} = \varphi_i^- \circ (\varphi_i^+)^{-1} = \mathrm{Id}_{\mathbb{B}^n}.$$

Thus, the collection of charts $\{(U_i^\pm, \varphi_i^\pm)\}$ is a smooth atlas, and so defines a smooth structure on $\mathbb{S}^n$. We call this its **standard smooth structure**.

Example 1.51

Example 1.5 (Projective Spaces). The ***n-dimensional real projective space***, denoted by $\mathbb{RP}^n$ (or sometimes just $\mathbb{P}^n$), is defined as the set of 1-dimensional linear subspaces of $\mathbb{R}^{n+1}$, with the quotient topology determined by the natural map $\pi : \mathbb{R}^{n+1} \smallsetminus \{0\} \to \mathbb{RP}^n$ sending each point $x \in \mathbb{R}^{n+1} \smallsetminus \{0\}$ to the subspace spanned by x. The 2-dimensional projective space $\mathbb{RP}^2$ is called the ***projective plane***. For any point $x \in \mathbb{R}^{n+1} \smallsetminus \{0\}$, let $[x] = \pi(x) \in \mathbb{RP}^n$ denote the line spanned by x.

For each $i = 1, \ldots, n+1$, let $\widetilde{U}_i \subseteq \mathbb{R}^{n+1} \smallsetminus \{0\}$ be the set where $x^i \neq 0$, and let $U_i = \pi\left(\widetilde{U}_i\right) \subseteq \mathbb{RP}^n$. Since $\widetilde{U}_i$ is a saturated open subset, U_i is open and $\pi|_{\widetilde{U}_i} : \widetilde{U}_i \to U_i$ is a quotient map (see Theorem A.27). Define a map $\varphi_i : U_i \to \mathbb{R}^n$ by

$$\varphi_i\left[x^1, \ldots, x^{n+1}\right] = \left(\frac{x^1}{x^i}, \ldots, \frac{x^{i-1}}{x^i}, \frac{x^{i+1}}{x^i}, \ldots, \frac{x^{n+1}}{x^i}\right).$$

This map is well defined because its value is unchanged by multiplying x by a nonzero constant. Because $\varphi_i \circ \pi$ is continuous, φ_i is continuous by the characteristic property of quotient maps (Theorem A.27). In fact, φ_i is a homeomorphism, because it has a continuous inverse given by

$$\varphi_i^{-1}\left(u^1, \ldots, u^n\right) = \left[u^1, \ldots, u^{i-1}, 1, u^i, \ldots, u^n\right],$$

as you can check. Geometrically, $\varphi([x]) = u$ means $(u, 1)$ is the point in $\mathbb{R}^{n+1}$ where the line $[x]$ intersects the affine hyperplane where $x^i = 1$ (Fig. 1.4). Because the sets $U_1, \ldots, U_{n+1}$ cover $\mathbb{RP}^n$, this shows that $\mathbb{RP}^n$ is locally Euclidean of dimension n. The Hausdorff and second-countability properties are left as exercises. //

Example 1.52

Let $[a^0, \dots, a^n]$ be homogeneous coordinates on the projective space $\mathbb{R}P^n$. Although a^0 is not a well-defined function on $\mathbb{R}P^n$, the condition $a^0 \neq 0$ is independent of the choice of a representative for $[a^0, \dots, a^n]$. Hence, the condition $a^0 \neq 0$ makes sense on $\mathbb{R}P^n$, and we may define

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$$U_0 := \{[a^0, \dots, a^n] \in \mathbb{R}P^n \mid a^0 \neq 0\}.$$

Similarly, for each $i = 1, \dots, n$, let

$$U_i := \{[a^0, \dots, a^n] \in \mathbb{R}P^n \mid a^i \neq 0\}.$$

Define

$$\phi_0 \colon U_0 \to \mathbb{R}^n$$

by

$$[a^0, \dots, a^n] \mapsto \left(\frac{a^1}{a^0}, \dots, \frac{a^n}{a^0}\right).$$

This map has a continuous inverse

$$(b^1, \dots, b^n) \mapsto [1, b^1, \dots, b^n]$$

and is therefore a homeomorphism. Similarly, there are homeomorphisms for each $i = 1, \dots, n$:

$$\phi_i \colon U_i \to \mathbb{R}^n,$$

$$[a^0, \dots, a^n] \mapsto \left(\frac{a^0}{a^i}, \dots, \widehat{\frac{a^i}{a^i}}, \dots, \frac{a^n}{a^i}\right),$$

where the caret sign $\widehat{}$ over a^i/a^i means that that entry is to be omitted. This proves that $\mathbb{R}P^n$ is locally Euclidean with the (U_i, ϕ_i) as charts.

Proposition 1.53

Let M and N be two smooth manifolds, then $M \times N$ admits a product smooth structure.

Proof. Suppose that X is equipped with an atlas $\mathcal{A} = \{(U_i, \varphi_i)\}$ and Y is equipped with an atlas $\mathcal{B} = \{(V_j, \psi_j)\}$. Then the collection

$$\{(U_i \times V_j, \varphi_i \times \psi_j)\}$$

is an atlas for $X \times Y$. Moreover, any two charts in it are smoothly compatible, since

$$(\varphi_i \times \psi_j) \circ (\varphi_{i'} \times \psi_{j'})^{-1} = (\varphi_i \circ \varphi_{i'}^{-1}) \times (\psi_j \circ \psi_{j'}^{-1}).$$

This further tells us the dimension is additive

$$\dim(X \times Y) = \dim X + \dim Y.$$

□

Example 1.54. The product of n-copies of S^1 is called the n-dimensional torus $T^n := S^1 \times \cdots \times S^1$.

Example 1.55. Note that the disjoint union of two topological spaces has a natural topology. The disjoint union of two smooth manifolds with the same dimension is also a smooth manifold with the same dimension.

2 Smooth maps

2.1 Smooth functions on manifolds

Definition 2.1. Suppose M is a smooth n-manifold, $k \geq 0$, and $f : M \to \mathbb{R}^k$ is any function. We say f is a **smooth function**

- if for every $p \in M$ there exists a smooth chart (U, φ) containing p;
- the composition $f \circ \varphi^{-1}$ is smooth on the open subset $\varphi(U) \subset \mathbb{R}^n$

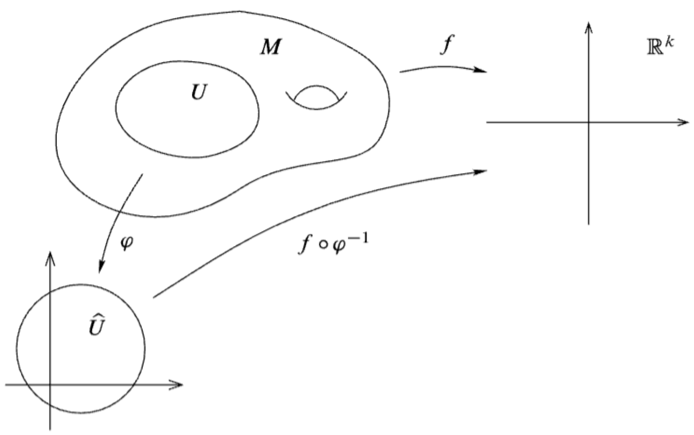

Figure 4: Definition of a smooth function

2.2 Smooth maps between manifolds

Definition 2.2. Let M and N be smooth manifolds, and let $F : M \to N$ be any map. We say that F is a **smooth map**

- if for evert $p \in M$, there exists smooth charts containing (U, φ) containing p and (V, ψ) containing $F(p)$ such that $F(U) \subseteq V$;
- the composite map $\psi \circ F \circ \varphi^{-1} : \varphi(U) \to \psi(V)$ is smooth. Figure 5 illustrates this definition.

> **Remark 2.3.** The definition of the smoothness of F does not depend on the charts (which are smoothly compatible).

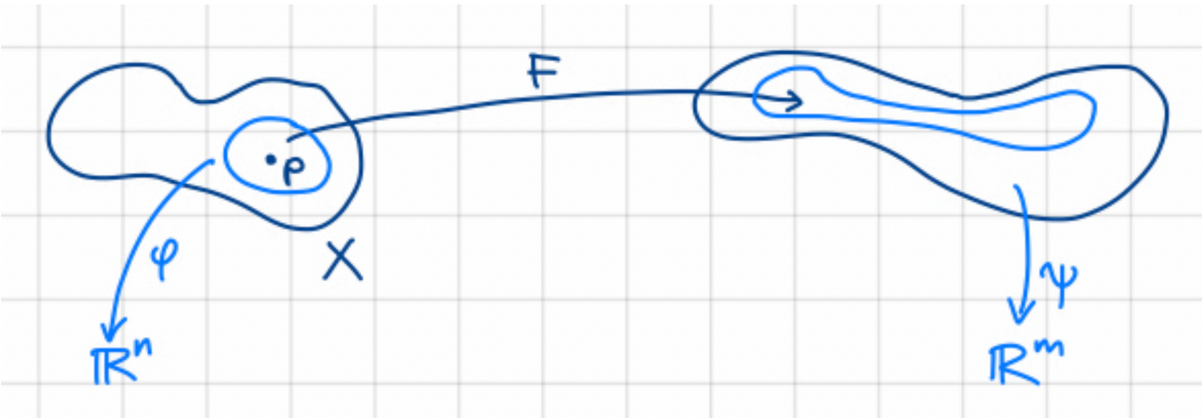

Figure 5: Smooth map between manifolds

Example 2.4. Some examples.

- It is clear that the function $F : \mathbb{R} \to \mathbb{R}$ where $F(x) = x$ is smooth by considering the two charts $(\mathbb{R}, \mathrm{id})$ and $(\mathbb{R}, \mathrm{id})$.
- Consider the manifold $\mathbb{R}$ equipped with two different atlases: $M = (\mathbb{R}, \mathcal{A} = (\mathbb{R}, \varphi = \mathrm{id}))$ and $N = (\mathbb{R}, \mathcal{B} = (\mathbb{R}, x^3))$. The map $F : M \to N$ given by $F(x) = x$ is smooth since $\psi \circ F \circ \varphi^{-1} = x^3$ is a smooth function.
- Non-example. Consider $G : N \to M$ given by $G(x) = x$ where M and N are defined the same as above. This function is not smooth since $\varphi \circ G \circ \psi^{-1} = x^{1/3}$ which is not smooth.

Definition 2.5. The space of smooth functions from M to N is written $C^\infty(M, N)$. In particular, $C^\infty(M, \mathbb{R}) = C^\infty(M)$.

Definition 2.6. If M and N are smooth manifolds, a **diffeomorphism** from M to N is a smooth bijective map $F : M \to N$ that has a smooth inverse. Two manifolds are called **diffeomorphic** if there is a diffeomorphism between them.

Proposition 2.7. Properties of diffeomorphisms.

1. Every composition of diffeomorphisms is a diffeomorphism.
2. Every finite product of diffeomorphism between smooth manifolds is a diffeomorphism.
3. Every diffeomorphism is a homeomorphism.
4. "Diffeomorphic" is an equivalence relation on the class of all smooth manifolds.

Example 2.8 (Classification of smooth manifolds up to diffeomorphism)
For $\mathbb{R}^n$:

1. If $n \neq 4$ there exist a unique smooth structure up to diffeomorphism.
2. $\mathbb{R}^4$ admits many smooth structures, no two of which are homeomorphic (Simon Donaldson, Michael Freedman 1984).

2.3 Gluing smooth maps

Lemma 2.9 (Gluing lemma for smooth maps)
Let M and N be smooth manifolds and let U, V be two open subsets of M such that $U \cup V = M$. Suppose we have two smooth maps

$$F_U : U \to N \quad \text{and} \quad F_V : V \to N$$

such that $F_U = F_V$ on $U \cap V$. Then there exists a smooth map $F : M \to N$ such that $F|_U = F_U$ and $F|_V = F_V$.

Proof. Set $F : M \to N$, we have that

$$F(x) = \begin{cases} F_U(x) & \text{if } x \in U \\ F_V(x) & \text{if } x \in V. \end{cases}$$

If $x \in U \cap V$ then $F_U(x) = F_V(x)$ so, it is a well-defined function. We prove F is continuous. Pick an open subset $B \subset N$ then

$$\begin{aligned} F^{-1}(B) &= (F^{-1}(B) \cap U) \cup (F^{-1}(B) \cap V) \\ &= F_U^{-1}(B) \cup F_V^{-1}(B), \end{aligned}$$

which is the union of open sets. To prove F is smooth we pick $x \in M$, since $M = U \cup V$ then x is in one of these sets. Say $x \in U$ then pick a chart W such that $W \subseteq U$ and $x \in W$ then restrict F to U which gives us F_U which is smooth. □

Example 2.10 (Necessity of open sets)
This example is to illustrate why we need open sets. Consider the manifold $\mathbb{R}$ and its subsets $U = [0, \infty)$ and $V = (-\infty, 0]$. Define $F_U(x) = x$ and $F_V(x) = -x$. By drawing this function we obtain $|x|$, but this function is not smooth.

2.4 Smooth functions on closed subsets

Note 2.11. A frequently used tool in topology is the gluing lemma, which shows how to construct continuous maps by "gluing together" maps defined on open or closed subsets. We have a version of the gluing lemma for smooth maps defined on open subsets, but we cannot expect to glue together smooth maps defined on closed subsets and obtain a smooth result.

Definition 2.12 (Smooth function on subsets)**.** Let M be a smooth manifold and $K \subset M$ a subset. A function $F : K \to \mathbb{R}$ is called **smooth** if for all $x \in K$ there exists an open subset $U_x \subset M$ containing x and a smooth function $\widetilde{F}_x : U_x \to \mathbb{R}$ such that $\widetilde{F}_x|_{U_x \cap K} = F$.

Example 2.13. A simple example of extending functions is to consider $\mathbb{R}$ as our manifold. Define your favourite smooth function defined on a closed set such as $K = (-\infty, 0] \cup [1, \infty]$. Clearly, there is a gap in the interval $(0, 1)$. To extend our function defined on K we can define a new function by gluing two smooth functions together.

Note 2.14. We will now prove the existence of such functions. There will be two types that we need to consider.

Lemma 2.15 (Cut-off function)**.** Given any real numbers r_1 and r_2 such that $r_1 < r_2$, there exists a smooth function $f : \mathbb{R} \to \mathbb{R}$ such that

1. $0 \leq f(x) \leq 1$ for all $x \in \mathbb{R}$.
2. $f(x) = 0$ when $x \geq r_2$.
3. $f(x) = 1$ when $x \leq r_1$.
4. $0 < f(x) < 1$ when $x \in (r_1, r_2)$.
5. f is non-decreasing.

We call such f a **cut-off function**.

Remark 2.16. This function is not unique.

Proof. Consider the function

$$g(x) = \begin{cases} e^{-1/x} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

and set

$$f(x) = \frac{g(x - r_1)}{g(x - r_1) + g(r_2 - x)}.$$

□

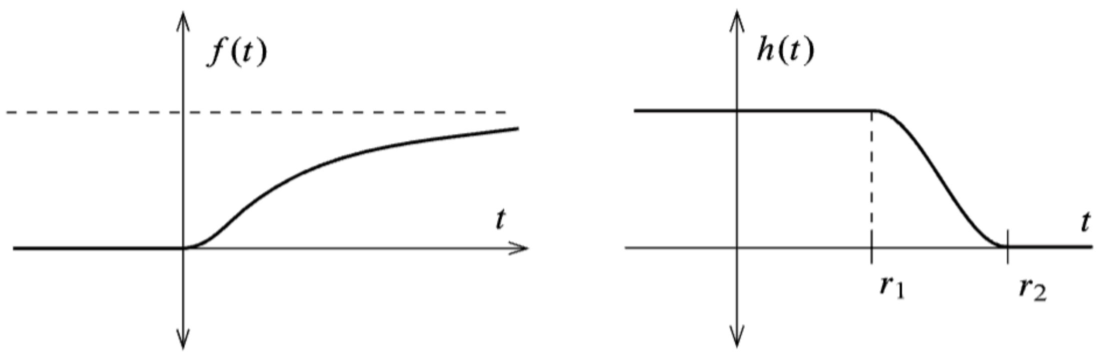

Figure 6: On the left we have $e^{1/x}$ and on the right a reflected about y-axis cut-off function.

Remark 2.17. Note that the product $f \cdot F$ of the cut-off function and any smooth function $F : \mathbb{R} \to \mathbb{R}$ is a smooth function such that

$$f \cdot F = \begin{cases} F & \text{if } x \leq r_1 \\ 0 & \text{if } x \geq r_2 \end{cases}$$

Definition 2.18. Let $r : \mathbb{R}^n \to \mathbb{R}$ be the **radius function**:

$$r(x) = \sqrt{x_1^2 + \cdots + x_n^2}.$$

An **open ball** of radius $c > 0$ centred at the origin is

$$B_c(0) = \{x \in \mathbb{R}^n : r(x) < c\}.$$

A **closed ball** of radius $c > 0$ centred at the origin is

$$\overline{B}_c(0) = \{x \in \mathbb{R}^n : r(x) \leq c\}.$$

Lemma 2.19 (Bump function)**.** Given any real positive real number $r_1 < r_2$, there exists a smooth function $h : \mathbb{R}^n \to \mathbb{R}$ such that

1. $0 < h(x) < 1$ when $r_1 < r(x) < r_2$.
2. $h(x) = 0$ when $x \notin B_{r_2}(0)$.
3. $h(x) = 1$ when $x \in \overline{B}_{r_1}(0)$.

Such function is called a **bump function**.

Note 2.20. It literally looks like a bump.

Proof. Set $h(x) = f(r(x))$ where f is the cut-off function. □

2.5 Partition of unity

Definition 2.21. If f is any real-valued or vector-valued function on a topological space M, the **support** of f, denoted by $\operatorname{supp}(f)$, is the closure of the set of points where f is non-zero:

$$\operatorname{supp}(f) = \text{Closure}(\{p \in M : f(p) \neq 0\}).$$

If $\operatorname{supp}(f)$ is contained in some set $U \subseteq M$, we say that f is **supported** in U.

Note 2.22. The support is a set where the function is not equal to zero.

Definition 2.23. Let M be a smooth manifold. Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ be an open cover of M. A **partition of unity** subordinate to the cover $\mathcal{U}$ is a collection of smooth function $\{f_\alpha : M \to \mathbb{R}\}_{\alpha \in A}$, such that

1. $0 \leq f_\alpha(p) \leq 1$ for all $\alpha \in A$ and all $p \in M$.
2. $\operatorname{supp}(f_\alpha) \subseteq U_\alpha$ for any $\alpha \in A$.
3. Every point $p \in M$ has a neighbourhood V_p such that only finitely many f_α are non-zero on V_p.
4. $\sum_{\alpha \in A} f_\alpha(p) = 1$ for all $p \in M$.

Example 2.24

Let $M = \mathbb{R}$ be our manifold with open cover $\mathcal{U} = \{U_1 = (-1, \infty), U_2 = (-\infty, 1)\}$. The idea is to use cut-off functions.

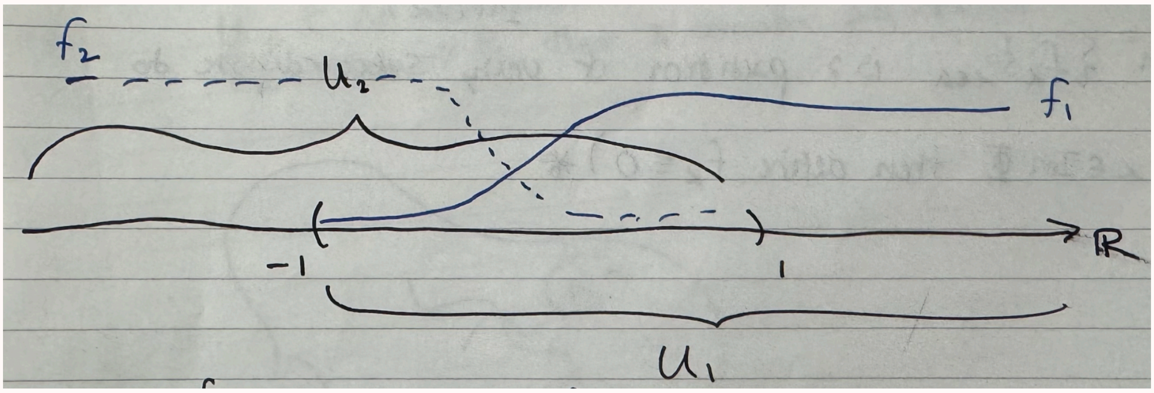

Figure 7: Partition of unity of $\mathbb{R}$.

Theorem 2.25 (Existence of partition of unity)

For any open cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ of a smooth manifold M, there exists a partition of unity subordinate to $\mathcal{U}$.

Proof. We will only prove the theorem when M is compact.
Any point $p \in M$ admits a coordinate chart (V_p, φ_p) such that $\varphi_p(V_p)$ is an open ball in $\mathbb{R}^n$. By shrinking V_p we can assume V_p is contained in some U_α. Note that $\{V_p\}$ is an open cover of M; by the compactness of M it admits to a finite subcover $\{V_{p_1}, V_{p2}, \ldots, V_{p_n}\}$. Now consider the collection of smooth functions $\{h \circ \varphi_{p_i}\}$ where h is the "bumped function" and φ_{p_i} is the homeomorphism of each chart containing p_i. We can define a smooth function $g = \sum_{i=1}^{n} h \circ \varphi_{p_i}$, note that $g > 0$ since each point p is contained in some V_{p_i}, which enables us to define

$$f_i = \frac{h \circ \varphi_{p_i}}{g},$$

which is a smooth function. By definition of f_i it follows that $\sum_i f_i = 1$. Since V_p is contained in some U_α we can choose a function $\Phi : \{1, 2, \ldots, n\} \to A$ such that $V_{p_i} \subseteq U_{\Phi(i)}$. Now for any $\alpha \in A$ we can define $f_\alpha = \sum_{\Phi(i)=\alpha} f_i$ and if $\alpha \in \mathrm{Im}\Phi$ then define $f_\alpha \equiv 0$. This collection $\{f_\alpha\}_{\alpha \in A}$ gives a partition of unity. □

2.5.1 Applications of partitions of unity

Lemma 2.26 (Extension of smooth functions)
Let M be a smooth manifold and let $K \subset M$ be a closed subset. For any smooth function $F : K \to \mathbb{R}$ there exists a smooth 'extension' $\widetilde{F}$, such that $\widetilde{F}|_K = F$.

Note 2.27. We are extending a smooth function F defined on a closed subset of the manifold to a globally smooth function.

Proof. Since $F : K \to \mathbb{R}$ is a smooth function then for any point $x \in K$ we can pick and open subset U_x containing x and a smooth function $G_x : U_x \to \mathbb{R}$ such that $G_x|_{U_x \cap K} = F$. Hence, the collection $\{U_x : x \in K\} \cup \{M \setminus K\}$ is an open cover of M. Let $\{h_x : x \in K\} \cup \{h_0\}$ be a partition of unity subordinate to the open cover, we define

$$\widetilde{F}(p) = \sum_{x \in K} G_x(p) \cdot h_x(p) \quad \text{for all } p \in M.$$

By definition the above sum is a finite for any point p thus, $\widetilde{F}$ is a well-defined smooth function. We now show that $\widetilde{F}|_K = F$. Pick a point $p \in K$ then

$$\begin{aligned}
\widetilde{F}(p) &= \sum_{x \in K} g_x(p) h_x(p) \\
&= h_0(p) + \sum_{x \in K} g_x(p) h_x(p) \\
&= g_x(p) \qquad \underbrace{\left[h_0(p) + \sum_{x \in K} h_x(p) \right]}_{=1 \text{ by the definition of a partition of unity}} \\
&= F(p).
\end{aligned}$$

□

Remark 2.28. If K was an open subset then we cannot obtain an open cover for M, and so we would not be able to apply the theory of partitions of unity.

Example 2.29. Let $M = \mathbb{R}$ and $U = (0, \infty)$ with $f(x) = \frac{1}{x}$. Clearly, f does not admit to any smooth extension.

Proposition 2.30 (Level sets of smooth funcions)
Let $K \subset \mathbb{R}^n$ be a closed subset, then there exists a function $f : \mathbb{R}^n \to \mathbb{R}$ such that

1. f is smooth,
2. $f \geq 0$,
3. $f^{-1}(0) = K$.

Note 2.31. This proposition states that every closed subset of a manifold can be expressed as a level set[a] of smooth real-valued function.

[a] The set of points in the domain of a function where the function is constant

Proof. Since $\mathbb{R}^n \setminus K$ is open, for any point $p \in \mathbb{R}^n \setminus K$ there exists an open ball $B_r(p) \subset \mathbb{R}^n \setminus K$ for some $0 < r < 1$. The collection of these open balls $\{B_r(p) : p \in \mathbb{R}^n \setminus K\}$ forms an open cover for $\mathbb{R}^n \setminus K$. Since $\mathbb{R}^n$ is second countable then so is $\mathbb{R}^n \setminus K$ which means the open cover $\{B_r(p) : p \in \mathbb{R}^n \setminus K\}$ admits to a countable subcover $\{B_{r_i}(p_i) : i \in \mathbb{N}\}$. Fix a bump function $h : \mathbb{R}^n \to \mathbb{R}$ such that

$$h(p) = \begin{cases} 0 & \text{if } p \notin B_1(0) \\ > 0 & \text{if } p \in B_1(0) \\ 1 & \text{if } p \in B_{\frac{1}{2}}(0). \end{cases}$$

For any $i \geq 1$ pick a constant C_i which is larger than all the partial derivative of h up to order i, and define $f : \mathbb{R}^n \to \mathbb{R}$ such that

$$f(x) = \sum_{i=1}^{\infty} \frac{(r_i)^i}{2^i \cdot C_i} h\left(\frac{p - x_i}{r_i}\right).$$

We claim that f is smooth, non-negative and $f^{-1}(0) = K$. $\square$

3 Tangent space

3.1 Partial derivatives

On a manifold M of dimension n, let (U, φ) be a chart and f a C^∞ function. As a function into $\mathbb{R}^n$, φ has n components $x^1, \ldots, x^n$. This means that if $r^1, \ldots, r^n$ are the standard coordinates on $\mathbb{R}^n$, then $x^i = r^i \circ \varphi$.

Note 3.1. The r^i are the i-th projection map, i.e. the pick out the i-th value.

Definition 3.2. For $p \in U$, we define the **partial derivative** $\partial f/\partial x^i$ of $f : M \to \mathbb{R}^n$ with respect to x^i at p to be

$$\left.\frac{\partial}{\partial x^i}\right|_p f := \frac{\partial f}{\partial x^i}(p) := \left.\frac{\partial (f \circ \varphi^{-1})}{\partial r^i}\right|_{\varphi(p)} = \left.\frac{\partial}{\partial r^i}\right|_{\varphi(p)} (f \circ \varphi^{-1}).$$

Note 3.3. Since the r^i pick the i-th coordinate, we have that for example in $\mathbb{R}^2$ $r^1(x,y) = x$ and $r^2(x,y) = y$. Thus, we are taking the partial derivative with respect to x and y.

Since $p = \varphi^{-1}(\varphi(p))$, this equation may be rewritten in the form

$$\frac{\partial f}{\partial x^i}\left(\varphi^{-1}(\varphi(p))\right) = \frac{\partial (f \circ \varphi^{-1})}{\partial r^i}(\varphi(p)).$$

Thus, as functions on $\varphi(U)$,

$$\frac{\partial f}{\partial x^i} \circ \varphi^{-1} = \frac{\partial (f \circ \varphi^{-1})}{\partial r^i}.$$

Proposition 3.4. The partial derivative $\partial f/\partial x^i$ is C^∞ on U because its pullback $(\partial f/\partial x^i) \circ \varphi^{-1}$ is C^∞ on $\varphi(U)$.

Note 3.5. The key idea is that although f is defined on a curved manifold M, we can only differentiate it meaningfully using the coordinates of a chart. So we pull back f to the Euclidean space via the chart φ, compute the partial derivative in $\mathbb{R}^n$, and interpret this as the partial derivative on the manifold.

3.2 Tangent vectors

Note 3.6. So far on a smooth manifold we have only defined smooth functions. We will utilise this to define 'tangent vectors' on a smooth manifold as the equivalent of a directional derivative.

Definition 3.7. Let M be a smooth manifold and $p \in M$ a point. A **tangent vector** at p is a map $v : C^\infty(M) \to \mathbb{R}$ which satisfies the following:

1. (Linearity) $v(\alpha f + \beta g) = \alpha v(f) + \beta f(g) + \beta v(g)$ for all $\alpha, \beta \in \mathbb{R}$ and $f, g \in C^\infty(M)$.
2. (Leibniz's rule) $v(f \cdot g) = f(p)v(g) + v(f)g(p)$ for all $f, g \in C^\infty(M)$.

The set of all tangent vectors of $C^\infty(M)$ at p is denoted T_pM. This is a vector space over $\mathbb{R}$ called the **tangent space to M at p**.

Note 3.8. You should visualise tangent vectors on M, as "arrows" that are tangent to M and whose base points are attached to M at the given point.

Remark 3.9. Recall that $C^\infty(X) = \{f : X \to \mathbb{R} : f \text{ is smooth}\}$.

Note 3.10. [Tu10, Definition 8.1] defines a tangent vector at a point p in M as a derivation at p.

Example 3.11. Let $M = \mathbb{R}$, for any $p \in \mathbb{R}$ we can set

$$v(f) := \frac{\mathrm{d}}{\mathrm{d}x} f \bigg|_p = f'(p).$$

Then v is a tangent vector at p. Similarly, the directional derivative of any smooth function on $\mathbb{R}^n$ is a tangent vector.

Example 3.12 ([Lee12, Corollary 3.3.])

Let $x_1, \dots, x_n$ be the coordinates of $\mathbb{R}^n$ then

$$\frac{\partial}{\partial x_1}\bigg|_p, \dots, \frac{\partial}{\partial x_n}\bigg|_p$$

denote the directional derivatives along $(1, 0, \dots, 0), (0, 1, \dots, 0)$ and $(0, 0, \dots, 1)$. The linear combination of $\left\{ \frac{\partial}{\partial x_1}\bigg|_p, \dots, \frac{\partial}{\partial x_n}\bigg|_p \right\}$ are tangent vectors. In fact, they form a basis of $T_p\mathbb{R}^n$ hence, $\dim(T_p\mathbb{R}^n) = n$.

Lemma 3.13 (Properties of tangent vectors on manifolds)**.** Suppose M is a smooth manifold, $p \in M, v \in T_pM$, and $f, g \in C^\infty(M)$.

1. If f is a constant function, then $v(f) = 0$.
2. If $f(p) = g(p) = 0$, then $v(fg) = 0$.

Proof. We prove each statement in turn.

1. By Leibniz's rule we can write

$$v(1) = v(1 \cdot 1) = 1 \cdot v(1) + v(1) \cdot 1 = 2v(1).$$

   Therefore, $v(1) = 0$. By the linearity of v we have that $\alpha v(1) = v(\alpha) = 0$ for all $\alpha \in \mathbb{R}$.

2. Trivial. □

Lemma 3.14

Let $f \in C^\infty(M)$ be such that there exists an open set U containing p with $f|_U = 0$ then $v(f) = 0$.

Proof. We consider a smaller open set around p, say $\mathcal{V}$ such that $\mathcal{V} \subseteq \text{Closure}(\mathcal{V}) \subset U$ (for example, we can choose U to be an open ball of arbitrary radius then pick V to be an open ball of smaller radius). Then we can choose a smooth function g such that $g|_{\mathcal{V}} = 0$ and $g|_{M \setminus \mathcal{U}} = 1$ (some sort of "bump" function). Note that function $f \cdot g = f$, then we can write

$$v(f) = v(f \cdot g) = f(p)v(g) + v(f)g(p) = 0$$

since f and g vanish at p. □

Corollary 3.15. If there exists an open set $U \subset M$ containing p such that $f = g$ on U then $v(f) = v(g)$.

Proof. Set $v(f - g)$ and use the proof of previous lemma. □

Example 3.16 (Necessity of open sets)
Consider the function $f(x) = x$ for $M = \mathbb{R}$. Clearly, f passes through 0 but $f'(x) = 1$ for all $x \in \mathbb{R}$, specifically at $x = 0$. Hence, why we need an open set around our point.

Lemma 3.17
Let $p \in \mathbb{R}^n$ and consider the map

$$\begin{aligned} F : T_p\mathbb{R}^n &\to \mathbb{R}^n \\ v &\mapsto (v(x_1), \ldots, v(x_n)) \end{aligned}$$

is a linear map. Moreover, it is an isomorphism hence, $\dim(T_p\mathbb{R}^n) = \dim(\mathbb{R}^n) = n$.

Proof. We prove linearity and bijectivity.

(Linearity). Let $\alpha, \beta \in \mathbb{R}$ then

$$\begin{aligned} F(\alpha v_1 + \beta v_2) &= ((\alpha v_1 + \beta v_2)(x_1), \cdots (\alpha v_1 + \beta v_2)(x_n)) \\ &= (\alpha v_1(x_1) + \beta v_2(x_1), \ldots, \alpha v_1(x_n) + \beta v(x_n)) \\ &= \alpha F(v_1) + \beta F(v_2). \end{aligned}$$

(Surjective). Let $(a_1, \ldots, a_n) \in \mathbb{R}^n$ set

$$v := a_1 \frac{\partial}{\partial x_1} + \cdots + a_n \frac{\partial}{\partial x_n}$$

then v is a tangent vector and $F(v) = (a_1, \ldots, a_n)$.

(Injective). Let $v \in T_pM$ with $F(v) = 0$. We will show that v is the zero tangent vector i.e. $v(f) = 0$ for any $f \in C^\infty(M)$. Consider the Taylor expansion of f at p:

$$f(x) = f(p) + \sum_{i=1}^{n} \frac{\partial}{\partial x_i} f(p(x_i - p_i)) + \sum_{i,j=1}^{n} (x_i - p_i) \int_0^1 (1-t) \frac{\partial^2}{\partial x_i \partial x_j} f(p + t(x-p))\, dt.$$

Apply v to both side we have that the first term is a constant function, hence it becomes zero under v. The second term also becomes zero since we assume $v(x_i) = 0$. The third term is a sum of products of $(x_i - p_i)$ and $(x_j - a_j)$ times an integral. Both of these vanish at p. Hence, each product becomes zero under v. By linearity, we have $v(f) = 0 + \sum 0 + \sum 0 = 0$. $\square$

3.3 The differential of a smooth map

Definition 3.18. If M and N are smooth manifolds and $F : M \to N$ is a smooth map, for each $p \in M$ we define a map

$$dF_p : T_pM \to T_{F(p)}N,$$

called the **differential of F at p** (Figure 8), as follows. Given $v \in T_pM$, we let

$$dF_p(v)(f) := v(f \circ F) \in \mathbb{R}$$

for all $f \in C^\infty(N)$.

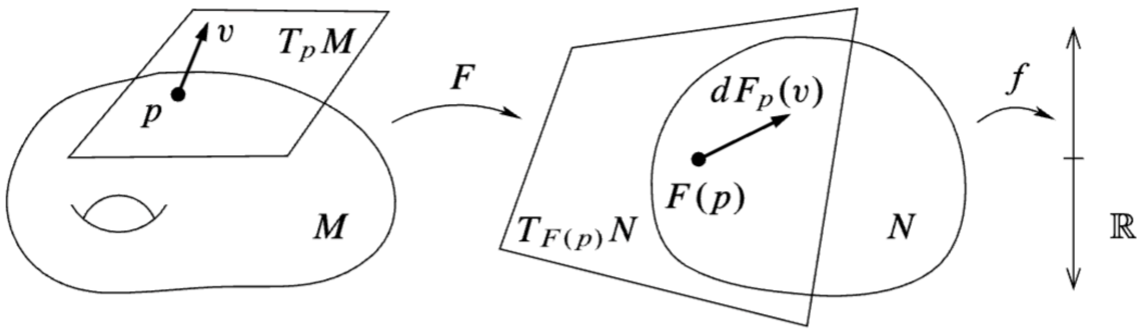

Figure 8: Illustration of differential

Remark 3.19. If $f \in C^\infty(N)$ then $f \circ F \in C^\infty(M)$ so, $v(f \circ F)$ makes sense.

Example 3.20 ([Lee12, Page 55])

In the case of $\mathbb{R}^n$ the differential is the Jacobian matrix at a point i.e. if $F : \mathbb{R}^n \to \mathbb{R}^m$ the Jacobian is a $n \times m$ matrix given by

$$J(F) = \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_m}{\partial x_1} & \cdots & \frac{\partial F_m}{\partial x_n} \end{pmatrix}$$

Corollary 3.21. The operator $dF_p(v) : C^\infty(N) \to \mathbb{R}$ is linear.

Proof. The map v is a tangent vector at $F(p)$. For any $f, g \in C^\infty(N)$ we have

$$\begin{aligned} dF_p(v)(fg) &= v((fg) \circ F) = v((f \circ F)(g \circ F)) \\ &= f \circ F(p) v(g \circ F) + g \circ F(p) v(f \circ F) \\ &= f(F(p)) dF_p(v)(g) + g(F(p)) dF_p(v)(f). \end{aligned}$$

□

Proposition 3.22 (Properties of differential)
Let M, N and P be smooth manifolds, let $F : M \to N$ and $G : N \to P$ be smooth maps and let $p \in M$.

1. $dF_p : T_pM \to T_{F(p)}N$ is linear.
2. $d(G \circ F)_p = dG_{F(p)} \circ dF_p : T_pM \to T_{G \circ F(p)}P$.
3. $d(\mathrm{id}_M)_p = \mathrm{id}_{T_pM} : T_pM \to T_pM$.
4. If F is a diffeomorphism then $dF_p : T_pM \to T_{F(p)}N$ is an isomorphism and $(dF_p)^{-1} = d(F^{-1})_{F(p)}$.

Proposition 3.23. Let M be a smooth manifold and let $U \subset M$ be an open subset, and let $\iota : U \to M$ be the inclusion map. The differential $d\iota_p : T_pU \to T_pM$ is an isomorphism for any $p \in U$.

Remark 3.24. This enables used to identify T_pU with T_pM.

Proof. We prove that $d\iota_p$ is a bijective map.

(Injective). Suppose $d\iota_p(v) = 0$, we need to show $v = 0 \in T_pU$. Suppose not, there exists $f \in C^\infty(U)$ with $v(f) \neq 0$. Pick a non-empty open subset V such that $p \in V \subset \mathrm{Closure}(V) \subset U$. Then f is also a smooth function on $\mathrm{Closure}(V)$. By the smooth extension lemma of smooth functions on closed sets, there is a smooth extension $\tilde{f} \in C^\infty(X)$ of $f|_{\mathrm{Closure}(V)}$. We get two smooth functions $f, \tilde{f}$ on U, which agree on V. Hence, we apply Corollary 3.15 to get

$$0 \neq v(f) = v(\tilde{f}|_U) = v(\tilde{f} \circ i) = d\iota_p(v)(\tilde{f}) = 0,$$

a contradiction.

(Surjective). For any $f \in C^\infty(U)$, we keep the above notation that $\tilde{f}$ is one smooth extension of $f|_{\mathrm{Closure}(V)}$. Next, for any $w \in T_pX$, we define a map

$$w' : C^\infty(U) \to \mathbb{R}, \quad w'(f) := w(\tilde{f}).$$

Again, by Corollary 3.15 we have w' is well-defined. More precisely, different choices of extensions give the same $w'(f)$. Moreover, one can directly check that w' is linear and satisfies the Leibniz rule. Hence, $w' \in T_pU$. Now for any $g \in C^\infty(X)$, we have

$$d\iota_p(w')(g) = w'(g \circ i) = w'(g|_U) = w(g).$$

The last equality holds because g is a smooth extension of $g|_{\mathrm{Closure}(V)}$. Hence, $d\iota_p(w') = w$.

□

Corollary 3.25. Let M be a smooth n-dimensional manifold. Let $p \in M$ then T_pM is an n-dimensional vector space.

Proof. Let (U, ϕ) be a smooth chart containing p. Then ϕ is a diffeomorphism and its differential is an isomorphism. Hence, we get two isomorphisms $d\iota_p : T_pU \to T_pX$ and $d\phi_p : T_pU \to T_{\phi(p)}\phi(U)$. We already know that $T_{\phi(p)}\phi(U)$ is an n-dimensional vector space. □

3.4 Computation in coordinates

Suppose M is a smooth manifold, and let (U, φ) be a smooth chart on M. Then φ is, in particular a diffeomorphism from U to an open subset $\varphi(U) \subseteq \mathbb{R}^n$. Combining previous propositions we see that

$$d\varphi_p : T_pM \to T_{\varphi(p)}\mathbb{R}^n$$

is an isomorphism. From a previous example, the set of tangent vectors

$$\left\{ \left.\frac{\partial}{\partial x_1}\right|_{\varphi(p)}, \dots, \left.\frac{\partial}{\partial x_n}\right|_{\varphi(p)} \right\}$$

forms a basis of $T_{\varphi(p)}\mathbb{R}^n$. Therefore, the pre-images of these vectors under the isomorphism $d\varphi_p$ form a basis for T_pM i.e.

$$\left\{ (d\varphi_p)^{-1} \left(\left.\frac{\partial}{\partial x_i}\right|_{\varphi(p)} \right) : i \in \{1, \dots, n\} \right\}$$

form a basis of T_pM.

Definition 3.26. We define the following notation for these basis vectors:

$$\left.\frac{\partial}{\partial x_i}\right|_p := \underbrace{(d\varphi_p)^{-1} \left(\left.\frac{\partial}{\partial x_i}\right|_{\varphi(p)} \right)}_{\text{pre-image}} = \underbrace{d\left(\varphi^{-1}\right)_{\varphi(p)} \left(\left.\frac{\partial}{\partial x_i}\right|_{\varphi(p)} \right)}_{\text{differential operator}}.$$

These vectors are called the **coordinate vectors at** p, associated with the given coordinate systems.

Note 3.27. So what we have is the following:

$$\begin{aligned} d\varphi_p : T_pM &\to T_{\varphi(p)}\mathbb{R}^n \\ \left.\frac{\partial}{\partial x_i}\right|_p &\mapsto \left.\frac{\partial}{\partial x_i}\right|_{\varphi(p)} \end{aligned}$$

Example 3.28

Let us see how this basis acts on $f \in C^\infty(U)$. Recall the differential

$$\begin{aligned} d\varphi_p &: T_pM \to T_{\varphi(p)}\mathbb{R}^n \\ d\varphi_p(v)(f) &= v(f \circ \varphi), \end{aligned}$$

for $v \in T_pM$ and $f \in C^\infty(U)$. Unwinding the definitions, we see that $\left.\frac{\partial}{\partial x_i}\right|_p$ acts on a function $f \in C^\infty(U)$ by

$$\begin{aligned} \left.\frac{\partial}{\partial x_i}\right|_p f &= d(\varphi^{-1})_{\varphi(p)}\left(\left.\frac{\partial}{\partial x_i}\right|_{\varphi(p)}\right) f \\ &= \left.\frac{\partial}{\partial x_i}\right|_{\varphi(p)} (f \circ \varphi^{-1}) \\ &= \frac{\partial(f \circ \varphi^{-1})}{\partial x_i}(\varphi(p)). \end{aligned}$$

In other words, $\left.\frac{\partial}{\partial x_i}\right|_p$ is just the tangent vector that takes the i-th partial derivative of f at p.

Example 3.29. In the case of the standard coordinates on $\mathbb{R}^n$, the vectors are literally the partial derivatives operators.

Corollary 3.30. A tangent vector $v \in T_pM$ can be written uniquely as a linear combination

$$v = \sum_{i=1}^n a_i \left.\frac{\partial}{\partial x_i}\right|_p,$$

with respect to a given chart.

3.4.1 The differential in coordinates

Note 3.31. We explore how the differentials look in coordinates.

Example 3.32 (Euclidean space)**.** Consider a smooth map $F : U \to V$, where $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ are open. For any $p \in U$, we will determine the matrix of $dF_p : T_p\mathbb{R}^n \to T_{F(p)}\mathbb{R}^m$ in terms of the standard coordinate bases. Using $(x_1, \ldots, x_n)$ to denote coordinates in the domain and $(y_1, \ldots, y_m)$ to denote those in the image. We use the chain rule to compute the action of dF_p on a typical basis vector as follows:

$$\begin{aligned} dF_p\left(\left.\frac{\partial}{\partial x_i}\right|_p\right) f = \left.\frac{\partial}{\partial x_i}\right|_p (f \circ F) &= \sum_{j=1}^m \frac{\partial f}{\partial y_i}(F(p))\frac{\partial F_j}{\partial x_i}(p) \\ &= \sum_{j=1}^m \left(\frac{\partial F_j}{\partial x_i}(p)\left.\frac{\partial}{\partial y_j}\right|_{F(p)}\right) f. \end{aligned}$$

Thus

$$dF_p\left(\frac{\partial}{\partial x_i}\bigg|_p\right) = \sum_{j=1}^{m} \frac{\partial F_j}{\partial x_i}(p)\frac{\partial}{\partial y_j}\bigg|_{F(p)}.$$

In other words, the matrix of dF_p in terms of the coordinate basis is the Jacobian matrix of F at p.

Example 3.33

(Local expression of differential from [Tu10, Section 8.5])

> **Remark 3.34.** F_* denotes the differential of F.

Given a smooth map $F : N \to M$ of manifolds and $p \in N$, let $(U, x^1, \ldots, x^n)$ be a chart about p in N and let $(V, y^1, \ldots, y^m)$ be a chart about $F(p)$ in M. We will find a local expression for the differential $F_{*,p} : T_pN \to T_{F(p)}M$ relative to the two charts.

By Proposition 8.9, $\left\{ \frac{\partial}{\partial x^j}\big|_p \right\}_{j=1}^{n}$ is a basis for T_pN and $\left\{ \frac{\partial}{\partial y^i}\Big|_{F(p)} \right\}_{i=1}^{m}$ is a basis for $T_{F(p)}M$. Therefore, the differential $F_* = F_{*,p}$ is completely determined by the numbers a^i_j such that

$$F_*\left(\frac{\partial}{\partial x^j}\bigg|_p\right) = \sum_{k=1}^{m} a^k_j \frac{\partial}{\partial y^k}\bigg|_{F(p)}, \qquad j = 1, \ldots, n.$$

Applying both sides to y^i, we find that

$$a^i_j = \left(\sum_{k=1}^{m} a^k_j \frac{\partial}{\partial y^k}\bigg|_{F(p)}\right) y^i = F_*\left(\frac{\partial}{\partial x^j}\bigg|_p\right) y^i = \frac{\partial}{\partial x^j}\bigg|_p (y^i \circ F) = \frac{\partial F^i}{\partial x^j}(p).$$

We state this result as a proposition.

Proposition 3.35. Given a smooth map $F : N \to M$ of manifolds and a point $p \in N$, let $(U, x^1, \ldots, x^n)$ and $(V, y^1, \ldots, y^m)$ be coordinate charts about p in N and $F(p)$ in M respectively. Relative to the bases $\left\{ \frac{\partial}{\partial x^j}\big|_p \right\}$ for T_pN and $\left\{ \frac{\partial}{\partial y^i}\Big|_{F(p)} \right\}$ for $T_{F(p)}M$, the differential $F_{*,p} : T_pN \to T_{F(p)}M$ is represented by the matrix $\left[\frac{\partial F^i}{\partial x^j}(p)\right]$, where $F^i = y^i \circ F$ is the ith component of F.

> **Remark 3.36.** In the context of manifolds, we denote the standard coordinates on $\mathbb{R}^n$ by $r^1, \ldots, r^n$. If $(U, \phi : U \to \mathbb{R}^n)$ is a chart of a manifold, we let $x^i = r^i \circ \phi$ be the ith component of ϕ and write $\phi = (x^1, \ldots, x^n)$ and $(U, \phi) = (U, x^1, \ldots, x^n)$. Thus, for $p \in U$, $(x^1(p), \ldots, x^n(p))$ is a point in $\mathbb{R}^n$. The functions $x^1, \ldots, x^n$ are called *coordinates* or *local coordinates* on U. By abuse of notation, we sometimes omit the p. So the notation $(x^1, \ldots, x^n)$ stands alternately for local coordinates on the open set U and for a point in $\mathbb{R}^n$. By a *chart* (U, ϕ) *about* p in a manifold M, we will mean a chart in the differentiable structure of M such that $p \in U$.

Example 3.37 ([Lee12, Page 62])**.** Now consider the more general case of a smooth map $F : M \to N$ between smooth maps. Choosing smooth coordinate charts (U, ϕ) for M

containing p and (V, ψ) for N containing $F(p)$, we obtain the coordinate representation

$$\widehat{F} = \psi \circ F \circ \phi^{-1} : \phi(U \cap F^{-1}(V)) \to \psi(V).$$

Now $d\widehat{F}_{\phi(p)}$ is represented with respect to the standard coordinate bases by the Jacobian matrix of $\widehat{F}$ at $\phi(p)$. Using the fact $F \circ \phi^{-1} = \psi^{-1} \circ \widehat{F}$, we have that

$$dF_p \left(\left.\frac{\partial}{\partial x_i}\right|_p \right) = \sum_{j=1}^{m} \frac{\partial \widehat{F}_j}{\partial x_i}(\phi(p)) \left.\frac{\partial}{\partial y_j}\right|_{F(p)}.$$

Example 3.38

Let $M = \mathbb{R}_t$ and $N = S^1 = \{(x, y) : x^2 + y^2 = 1\} \subset \mathbb{R}^2$. Consider the map

$$\begin{aligned} F : M &\to N \\ t &\mapsto (\cos t, \sin t). \end{aligned}$$

Compute $dF_{\frac{\pi}{6}}$.

Solution. We begin by observing that $dF_{\frac{\pi}{6}} : T_{\frac{\pi}{6}}\mathbb{R} \to T_{F\left(\frac{\pi}{6}\right)}S^1$. To proceed with the computation, we need to choose a coordinate chart for N. For the computation we do not need a coordinate map ϕ for M, as by the definition (from above) this will be 'cancelled out'. For this example, we will perform the calculation using two different charts of N. Let $V = \{(x, y) \mid x > 0, y > 0\}$, and consider the following charts.

- (V, ψ_x) where $\psi_x(x, y) = x$.

$$\begin{aligned} dF_{\frac{\pi}{6}} \left(\left.\frac{\partial}{\partial t}\right|_{\frac{\pi}{6}} \right) &= \sum_{j=1}^{2} \left.\frac{\partial \widehat{F}_j}{\partial t}\right|_{\frac{\pi}{6}} \left.\frac{\partial}{\partial y_j}\right|_{F\left(\frac{\pi}{6}\right)} \\ &= \left.\frac{\partial(\psi_x \circ F)}{\partial t}\right|_{\frac{\pi}{6}} \left.\frac{\partial}{\partial x}\right|_{F\left(\frac{\pi}{6}\right)} \\ &= \frac{\partial}{\partial t}[\cos t]\big|_{\frac{\pi}{6}} \left.\frac{\partial}{\partial x}\right|_{F\left(\frac{\pi}{6}\right)} \\ &= -\sin\left(\frac{\pi}{6}\right) \frac{\partial}{\partial x} \end{aligned}$$

  Notice that in this case $\widehat{F}_2 = 0$.

- (V, ϕ_y) where $\psi_y(x, y) = y$.

$$\begin{aligned} dF_{\frac{\pi}{6}} \left(\left.\frac{\partial}{\partial t}\right|_{\frac{\pi}{6}} \right) &= \left.\frac{\partial(\psi_y \circ F)}{\partial t}\right|_{\frac{\pi}{6}} \left.\frac{\partial}{\partial y}\right|_{F\left(\frac{\pi}{6}\right)} \\ &= \cos\left(\frac{\pi}{6}\right) \frac{\partial}{\partial y} \end{aligned}$$

Note 3.39. We use the notation $\mathbb{R}_t$ to mean that the parameter for this space is t.

3.4.2 Change of coordinates

Proposition 3.40 ([Lee12, Page 63–64])
Suppose (U,ϕ) and (V,ψ) are two smooth charts on a M, and $p \in U \cap V$. Denote the coordinate functions of ϕ by x_i and those of ψ by $\widetilde{x}_i$. Any tangent vector at p can be represented with respect to either basis $\left(\frac{\partial}{\partial x_i}\bigg|_p\right)$ or $\left(\frac{\partial}{\partial \widetilde{x}_j}\bigg|_p\right)$. These two representations are related as follows:

$$\frac{\partial}{\partial x_i}\bigg|_p = \sum_{j=1}^{m} \frac{\partial \widetilde{x}_j}{\partial x_i}(\phi(p)) \frac{\partial}{\partial \widetilde{x}_j}\bigg|_p .$$

Note 3.41. For a chart (U,ϕ) where $\phi(p) = (x_1(p), \ldots, x_n(p))$ the x_i are what we call the **coordinate functions**.

Example 3.42 ([Lee12, Example 3.16.])
The transition map between polar coordinates and standard coordinates in suitable open subsets of the plane is given by $(x, y) = (r\cos\theta, r\sin\theta)$. Let $p \in \mathbb{R}^2$ whose polar coordinates representation is $(r, \theta) = \left(2, \frac{\pi}{2}\right)$, and let $v \in T_p\mathbb{R}^2$ be the tangent vector whose polar coordinates representation is

$$v = 3\frac{\partial}{\partial r}\bigg|_p - \frac{\partial}{\partial \theta}\bigg|_p.$$

Let us apply the proposition above, in this case the x_i are r and θ; the $\widetilde{x}_j$ are x and y. Using the proposition

$$\begin{aligned}\frac{\partial}{\partial r}\bigg|_p &= \frac{\partial}{\partial r}\left[r\cos\left(\frac{\pi}{2}\right)\right]\frac{\partial}{\partial x}\bigg|_p + \frac{\partial}{\partial r}\left[r\sin\left(\frac{\pi}{2}\right)\right]\frac{\partial}{\partial y}\bigg|_p \\ &= \cos\left(\frac{\pi}{2}\right)\frac{\partial}{\partial x}\bigg|_p + \sin\left(\frac{\pi}{2}\right)\frac{\partial}{\partial y}\bigg|_p \\ &= \frac{\partial}{\partial y}\bigg|_p.\end{aligned}$$

Similarly,

$$\begin{aligned}\frac{\partial}{\partial \theta}\bigg|_p &= \frac{\partial}{\partial \theta}\left[r\cos\theta\right]\bigg|_{\left(2,\frac{\pi}{2}\right)}\frac{\partial}{\partial x}\bigg|_p + \frac{\partial}{\partial \theta}\left[r\sin\theta\right]\bigg|_{\left(2,\frac{\pi}{2}\right)}\frac{\partial}{\partial y}\bigg|_p \\ &= -2\sin\left(\frac{\pi}{2}\right)\frac{\partial}{\partial x}\bigg|_p + 2\cos\left(\frac{\pi}{2}\right)\frac{\partial}{\partial y}\bigg|_p \\ &= -2\frac{\partial}{\partial x}\bigg|_p.\end{aligned}$$

Therefore, v has the following coordinate representation in standard coordinates:

$$v = 3\frac{\partial}{\partial y}\bigg|_p + 2\frac{\partial}{\partial x}\bigg|_p.$$

3.5 Velocity vectors of curves

Definition 3.43. If M is a smooth manifold, we defien a **curve in** M to be a smooth map $\gamma : (-1, 1) \to M$.

> **Remark 3.44.** In [Lee12, Page 68] they define a curve to be $\gamma : I \to M$ where $I \subset \mathbb{R}$.

Definition 3.45. Given a smooth curve $\gamma : (-1, 1) \to M$ and $t_0 \in (-1, 1)$, we define the **velocity** of γ to be the vector

$$\gamma'(t) = v_\gamma = d\gamma\left(\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0}\right) \in T_{\gamma(t_0)}M$$

where $\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}$ is the standard coordinate basis vector in $T_0\mathbb{R}$. (As in ordinary calculus, it is customary to use $\frac{\mathrm{d}}{\mathrm{d}t}$ instead of $\frac{\partial}{\partial t}$ when the manifold in 1-dimensional).

Proposition 3.46

The tangent vector v_γ acts on functions by

$$\begin{aligned}\gamma'(t)(f) = v_\gamma(f) &= d\gamma\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\right) f \\ &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t_0} (f \circ \gamma) \\ &= \frac{\mathrm{d}(f\circ\gamma)}{\mathrm{d}t}(t_0) \\ &= (f\circ\gamma)'(t_0).\end{aligned}$$

Proof. Definition chasing. □

Example 3.47

In general the f we use is the map obtained from the chart, i.e. φ.

Example 3.48

Define the curve $\gamma(t) = (t^2, t^3)$. It follows that

$$\gamma'(t) = d\gamma\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) = 2t\frac{\partial}{\partial x} + 3t\frac{\partial}{\partial y}.$$

Example 3.49

Proposition 8.15 (Velocity of a curve in local coordinates). *Let $c\colon\,]a,b[\to M$ be a smooth curve, and let $(U, x^1, \dots, x^n)$ be a coordinate chart about $c(t)$. Write $c^i = x^i \circ c$ for the ith component of c in the chart. Then $c'(t)$ is given by*

$$c'(t) = \sum_{i=1}^{n} \dot{c}^i(t) \left.\frac{\partial}{\partial x^i}\right|_{c(t)}.$$

Thus, relative to the basis $\{\partial/\partial x^i|_p\}$ for $T_{c(t)}M$, the velocity $c'(t)$ is represented by the column vector

$$\begin{bmatrix} \dot{c}^1(t) \\ \vdots \\ \dot{c}^n(t) \end{bmatrix}.$$

Proof. Problem 8.5. □

3.6 The tangent bundle

Note 3.50. Often it is useful to consider the set of all tangent vectors at all points of a manifold.

Definition 3.51. Given a smooth manifold M, we define the **tangent bundle** of M, to be the disjoint union of the tangent spaces at all points of M:

$$TM = \bigsqcup_{p \in M} T_p M.$$

We usually write an element of this disjoint union as an ordered pair (p, v) with $p \in M$ and $v \in T_p M$. The tangent bundle comes equipped with a natural **projection map** $\pi : TM \to M$ such that $\pi(p, v) = p$.

Example 3.52. Let M be a 1-dimensional smooth manifold i.e. a line. At every point $p \in M$ there exists tangent. The tangent bundle is the union of these spaces.

Proposition 3.53

For any smooth n-smooth manifold M, the tangent bundle TM has a natural topology and smooth structure such that TM is a smooth $2n$-dimensional manifold. With respect to this structure, the projection $\pi : TM \to M$ is smooth.

Proof. We begin by noting that the tangent bundle is equipped with a natural projection map

$$\begin{aligned} \pi : TM &\to M \\ (p, v) &\mapsto p. \end{aligned}$$

Now we define the maps that will become our smooth charts. Given any smooth chart (U, ϕ) for M, note that $\pi^{-1}(U) \subseteq TM$ is the set of all tangent vectors to M at all points of U. Let $(x_1, \ldots, x_n)$ denote the coordinate functions of ϕ, and define a map

$$\begin{aligned} \widetilde{\phi} &: \pi^{-1}(U) \to \mathbb{R}^{2n} \\ \widetilde{\phi}\left(\sum_{i=1}^{n} v_i \frac{\partial}{\partial x_i}\bigg|_p \right) &= (x_1(p), \ldots, x_n(p), v_1, \ldots, v_n), \end{aligned}$$

as illustrated in Figure 9.

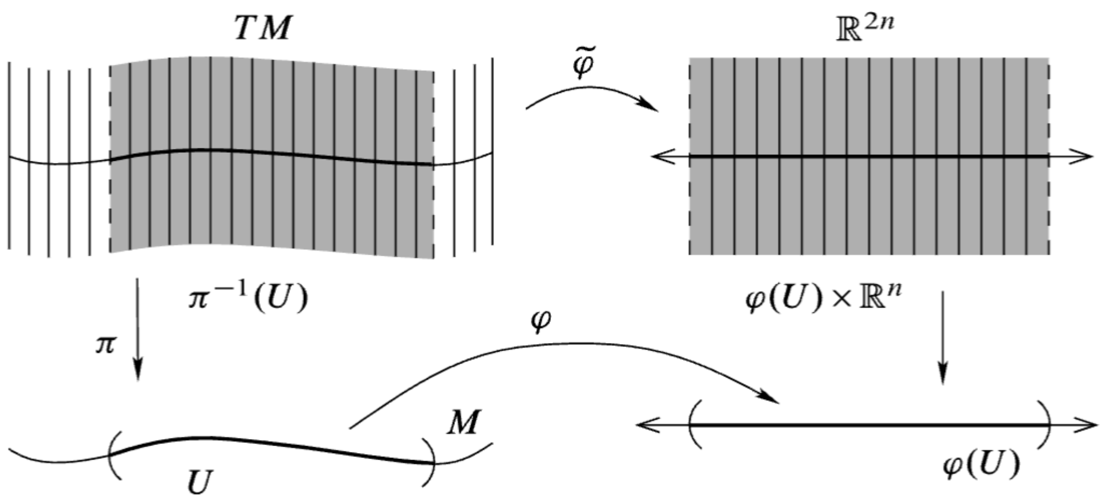

Figure 9: Coordinates for the tangent bundle [Lee12, Fig.3.8]

The image of $\widetilde{\phi}$ is $\phi(U) \times \mathbb{R}^n$, which is an open subset of $\mathbb{R}^{2n}$. Moreover, the map $\widetilde{\phi}$ is bijective. We define a subset of TM to be open if it is of the form $\widetilde{\phi}^{-1}(V)$ for some open subset $V \subset \mathbb{R}^{2n}$. This gives a topology on TM.
Next we check the charts are smooth. Let (U, ϕ) and (V, ψ) be two smooth charts on M and let $(\pi^{-1}(U), \widetilde{\phi})$ and $(\pi^{-1}(V), \widetilde{\psi})$ be the corresponding charts on TM. The transition map

$$\begin{aligned}\widetilde{\psi} \circ \widetilde{\phi}^{-1} &: \phi(U \cap V) \times \mathbb{R}^n \to \psi(U \cap V) \times \mathbb{R}^n \\ &(x_1, \ldots, x_n, v_1, \ldots, v_n) \mapsto \left(\widetilde{x}_1(x), \ldots, \widetilde{x}_n(x), \sum_{j=1}^n \frac{\partial \widetilde{x}_1}{\partial x_j}(x) v_j, \ldots, \sum_{j=1}^n \frac{\partial \widetilde{x}_n}{\partial x_j}(x) v_j \right),\end{aligned}$$

is clearly smooth. □

Example 3.54

In the special case $M = \mathbb{R}^n$ we see that the tangent bundle is just the Cartesian product of $\mathbb{R}^n$ with itself:

$$T\mathbb{R}^n = \bigsqcup_{a \in \mathbb{R}^n} T_a\mathbb{R}^n \cong \bigsqcup_{a \in \mathbb{R}^n} \mathbb{R}^n_a = \bigsqcup_{a \in \mathbb{R}^n} \{a\} \times \mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n.$$

Corollary 3.55

If M is a smooth n-dimensional manifold such that M can be covered by a single smooth chart, then TM is diffeomorphic to $M \times \mathbb{R}^n$.

Proof. If (U, ϕ) is the single global chart for M, then ϕ is, in particular, a diffeomorphism from $U = M$ to an open subset $\widetilde{U} \subseteq \mathbb{R}^n$. The proof of the previous proposition showed that the natural coordinate $\widetilde{\phi}$ is a bijection from TM ot $\widetilde{U} \times \mathbb{R}^n$, and the smooth structure on TM is defined essentially by declaring $\widetilde{\phi}$ to be a diffeomorphism. □

Definition 3.56. The tangent bundle TM of M is called **trivial** if TM is diffeomorphic to $M \times \mathbb{R}^n$.

Example 3.57. The unit circle has a trivial tangent bundle.

3.6.1 The total differential

Definition 3.58. Let $F : M \to N$ be a smooth map. The **total differential** $dF : TM \to TN$ is defined as the map whose restriction to any $p \in M$ is dF_p.

$$\begin{aligned} dF : TM &\to TN \\ (p, v) &\mapsto (F(p), dF_p(v)). \end{aligned}$$

The total differential is also called the **tangent map** of F.

Proposition 3.59. If $F : M \to N$ is a smooth map, then its total differential $dF : TM \to TN$ is a smooth map.

Corollary 3.60 (Properties of the total differential)
Suppose $F : M \to N$ and $G : N \to P$ are smooth maps.

1. $d(G \circ F) = dG \circ dF$.
2. $d(\mathrm{id}_M) = \mathrm{id}_{TM}$.
3. If F is a diffeomorphism, then $dF : TM \to TN$ is also a diffeomorphism, and $(dF)^{-1} = d(F^{-1})$.

4 Vector fields

Definition 4.1. If M is a smooth manifold, a **vector field** on M is a smooth map $\sigma : M \to TM$ with the property that

$$\pi \circ \sigma = \mathrm{id}_M,$$

where $\pi : TM \to M$ is the projection map.

Example 4.2

We can visualise a vector field on M in the same way we visualise vector fields in Euclidean space: as an arrow attached to each point of M, chosen to be tangent to M and vary continuously from point to point (Figure 10).

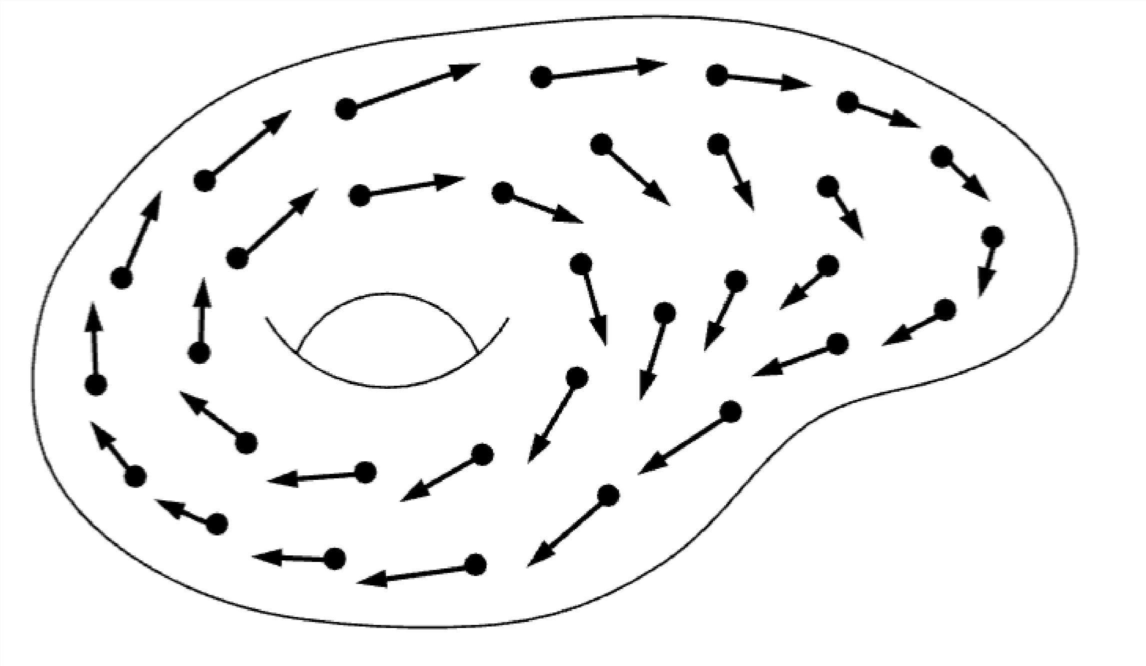

Figure 10: A vector field [Lee12, Fig. 8.1]

Example 4.3 (Coordinate vector fields[Lee12, Example 8.2.]). If (U, ϕ) is any smooth chart on M with coordinates x_i the assignment

$$p \mapsto \left.\frac{\partial}{\partial x_i}\right|_p$$

determines a vector field on U.

Example 4.4 (The Euler vector field [Lee12, Example 8.3.]). The vector field V on $\mathbb{R}^n$ whose value at $x \in \mathbb{R}^n$ is

$$V_x = x_1 \left.\frac{\partial}{\partial x_1}\right|_x + \cdots + x_n \left.\frac{\partial}{\partial x_n}\right|_x$$

is smooth because its coordinate functions are linear. It vanishes at the origin, and points radially outward everywhere else.

Definition 4.5. Let M be a smooth manifold, we use $\mathfrak{X}(M)$ to denote teh set of all smooth vector fields on M.

Proposition 4.6. $\mathfrak{X}(M)$ is a vector space, of infinite dimension.

Proof. Addition is pointwise and the zero element of this vector spcae is the zero vector field, whose value at each $p \in M$ is $0 \in T_pM$. In addition, smooth vector fields can be multplied by smooth real valued functions $f \in C^\infty(M)$. □

Example 4.7

Proposition 8.19. *Suppose M and N are smooth manifolds with or without boundary, and $F: M \to N$ is a diffeomorphism. For every $X \in \mathfrak{X}(M)$, there is a unique smooth vector field on N that is F-related to X.*

Proof. For $Y \in \mathfrak{X}(N)$ to be F-related to X means that $dF_p(X_p) = Y_{F(p)}$ for every $p \in M$. If F is a diffeomorphism, therefore, we *define* Y by

$$Y_q = dF_{F^{-1}(q)}\big(X_{F^{-1}(q)}\big).$$

It is clear that Y, so defined, is the unique (rough) vector field that is F-related to X. Note that $Y: N \to TN$ is the composition of the following smooth maps:

$$N \xrightarrow{F^{-1}} M \xrightarrow{X} TM \xrightarrow{dF} TN.$$

It follows that Y is smooth. □

In the situation of the preceding proposition we denote the unique vector field that is F-related to X by F_*X, and call it the ***pushforward of X by F***. Remember, it is only when F is a diffeomorphism that F_*X is defined. The proof of Proposition 8.19 shows that F_*X is defined explicitly by the formula

$$(F_*X)_q = dF_{F^{-1}(q)}\big(X_{F^{-1}(q)}\big). \tag{8.7}$$

As long as the inverse map F^{-1} can be computed explicitly, the pushforward of a vector field can be computed directly from this formula.

Example 8.20 (Computing the Pushforward of a Vector Field). Let M and N be the following open submanifolds of $\mathbb{R}^2$:

$$M = \big\{(x,y) : y > 0 \text{ and } x + y > 0\big\},$$
$$N = \big\{(u,v) : u > 0 \text{ and } v > 0\big\},$$

and define $F: M \to N$ by $F(x,y) = (x + y,\ x/y + 1)$. Then F is a diffeomorphism because its inverse is easily computed: just solve $(u,v) = (x + y,\ x/y + 1)$ for x and y to obtain the formula $(x,y) = F^{-1}(u,v) = (u - u/v,\ u/v)$. Let us

Example 4.8

compute the pushforward F_*X, where X is the following smooth vector field on M:

$$X_{(x,y)} = y^2 \frac{\partial}{\partial x}\bigg|_{(x,y)}.$$

The differential of F at a point $(x, y) \in M$ is represented by its Jacobian matrix,

$$DF(x, y) = \begin{pmatrix} 1 & 1 \\ \frac{1}{y} & -\frac{x}{y^2} \end{pmatrix},$$

and thus $dF_{F^{-1}(u,v)}$ is represented by the matrix

$$DF\left(u - \frac{u}{v}, \frac{u}{v}\right) = \begin{pmatrix} 1 & 1 \\ \frac{v}{u} & \frac{v - v^2}{u} \end{pmatrix}.$$

For any $(u, v) \in N$,

$$X_{F^{-1}(u,v)} = \frac{u^2}{v^2} \frac{\partial}{\partial x}\bigg|_{F^{-1}(u,v)}.$$

Therefore, applying (8.7) with $p = (u, v)$ yields the formula for F_*X:

$$(F_*X)_{(u,v)} = \frac{u^2}{v^2} \frac{\partial}{\partial u}\bigg|_{(u,v)} + \frac{u}{v} \frac{\partial}{\partial v}\bigg|_{(u,v)}. \qquad //$$

The next corollary follows directly from Proposition 8.16.

Corollary 8.21. *Suppose $F : M \to N$ is a diffeomorphism and $X \in \mathfrak{X}(M)$. For any $f \in C^\infty(N)$,*

$$\big((F_*X)f\big) \circ F = X(f \circ F). \qquad \square$$

4.1 Flow

Definition 4.9. Let M be a smooth manifold. A **flow** on M is a smooth map $\theta : \mathbb{R} \times M \to M$ satisfying the following properties for all $t_1, t_2 \in \mathbb{R}$ and $p \in M$:

1. $\theta(0, p) = p$, and
2. $\theta(t_1 + t_2, p) = \theta(t_1, \theta(t_2, p))$.

Example 4.10

Some examples and non-example of flows.

- The map $\theta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by $\theta(t, p) = p + t$ is a flow on $\mathbb{R}$.
- The map $\theta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by $\theta(t, p) = e^t p$ is a flow on $\mathbb{R}$.
- $\theta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by $\theta(t, p) = p + t^2$ is NOT a flow on $\mathbb{R}$, as it fails to satisfy the second condition.

Proposition 4.11

Any flow on a smooth manifold M determines a smooth vector field on M.

Proof. Let θ be a flow on M. For any point $p \in M$ and any $f \in C^\infty(M)$ we define

$$\sigma(p)(f) = \frac{\mathrm{d}}{\mathrm{d}t}(f \circ \theta(t, p))\bigg|_{t=0}.$$

This is a smooth vector field on M since θ depends on p smoothly. □

Example 4.12

The map $\theta : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ defined by $\theta(t, x, y) = (e^t x, y + t)$ is a flow on $\mathbb{R}^2$. The vector field determined by θ is

$$\sigma(x, y)(f) = \frac{\mathrm{d}}{\mathrm{d}t} f(e^t x, y + t)\bigg|_{t=0}.$$

Recall the for a function $f(u(t), v(t))$ is given by

$$\frac{\mathrm{d}}{\mathrm{d}t} f = \frac{\partial f}{\partial u}\frac{\mathrm{d}u}{\mathrm{d}t} + \frac{\partial f}{\partial v}\frac{\mathrm{d}v}{\mathrm{d}t}.$$

Now in this case $u(t) = e^t x$ and $v(t) = y + t$. Substituting back we have that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} f(e^t x, y + t)\bigg|_{t=0} &= e^0 x \frac{\partial f}{\partial u} + \frac{\partial f}{\partial v} \\
&= x \frac{\partial f}{\partial u} + \frac{\partial f}{\partial v} \\
&= \left(x \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) f,
\end{aligned}$$

where $u = x$ and $v = y$ at $t = 0$. Therefore, the vector field is

$$\sigma(x, y) = x \frac{\partial}{\partial x} + \frac{\partial}{\partial y}.$$

4.2 Integral curves

Note 4.13. Given a smooth curve on a smooth manifold M we can work out its velocity vector in $T_{\gamma(0)}M$. In this section we describe a way to work backwayrds: given a tangent vector at each point, we seek a curve whose velocity at each point is equal to the given vector there.

Definition 4.14. If σ is a vector field on M, an **integral curve of** M is a curve $\gamma : (a, b) \to M$ that satisfies the following differential equation

$$\gamma'(t) = \sigma(\gamma(t))$$

for all $t \in (a, b)$.

Example 4.15. A general smooth vector field on $\mathbb{R}^2$ is

$$\sigma(x, y) = f(x, y)\frac{\partial}{\partial x} + g(x, y)\frac{\partial}{\partial y}$$

for some smooth function f, g. Suppose $\gamma(t) = (\gamma_1(t), \gamma_2(t))$ is an integral curve of σ. Then, we have that

$$\begin{cases} \gamma_1'(t) & = f(\gamma_1(t), \gamma_2(t)) \\ \gamma_2'(t) & = g(\gamma_1, (t), \gamma_2(t)). \end{cases}$$

This is a system of ODEs. By the theory of ODEs, there always exists some solution $\gamma(t)$ with $t \in (-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$ and this solution is unique.

Lemma 4.16. Let M be a smooth manifold and let σ be a smooth vector field on M. For any $p \in M$ there exists $\varepsilon > 0$ and an integral curve

$$\begin{cases} \gamma : (-\varepsilon, \varepsilon) \to M \\ \gamma(0) = p \end{cases}$$

Two such integral curves are the same.

Note 4.17. We formalised the discussion above.

Example 4.18. On $\mathbb{R}^n$ vector fields generally look like

$$\sigma(x_1, \ldots, x_n) = f_1(x_1, \ldots, x_n)\frac{\partial}{\partial x_1} + \cdots f_n(x_1, \ldots, x_n)\frac{\partial}{\partial x_n}.$$

We consider the $\mathbb{R}^2$ case. Namely, when

$$\sigma(x_1, x_2) = f_1(x_1, x_2)\frac{\partial}{\partial x_1} + f_2(x_1, x_2)\frac{\partial}{\partial x_2}.$$

We want to find an integral curve $\gamma(t) = (x_1(t), x_2(t))$. Recall, that

$$\begin{aligned} \gamma'(t) &= dv_\gamma\left(\frac{\partial}{\partial t}\right) \\ &= A\frac{\partial}{\partial x_1} + B\frac{\partial}{\partial x_2}. \end{aligned}$$

To figure out A and B we apply the operator x_1 and x_2 to obtain:

$$\begin{aligned} dv_\gamma\left(\frac{\partial}{\partial t}\right)(x_1) &= \frac{\partial}{\partial t}(x_1 \circ \gamma) \\ &= \frac{\partial}{\partial t}(x_1(t)) \\ &= x_1'(t). \end{aligned}$$

Hence,

$$\gamma'(t) = x_1' \frac{\partial}{\partial x_1} + x_2' \frac{\partial}{\partial x_2}.$$

Comparing this to $\gamma'(t) = \sigma(\gamma(t))$ we have the following system of ODEs

$$\begin{cases} x_1'(t) &= f_1(x_1(t), y_2(t)) \\ x_2'(t) &= f_2(x_1(t), y_2(t)). \end{cases}$$

Example 4.19

Consider the manifold $\mathbb{R}$ and a vector field $\sigma(x) = x^2 \dfrac{\partial}{\partial x}$. Find an integral such that $\gamma(0) = 1$.

Solution. We want a integral curve i.e. a curve $\gamma(t)$ such that

$$\begin{aligned} \gamma'(t) &= \sigma(\gamma(t)) \\ &= \gamma^2(t). \end{aligned}$$

Using standard theory of ODE, we obtain that $\gamma(t) = \frac{1}{1-t}$ for $t \in (-1, 1)$.

5 Submersions, Immersions, and Embeddings

Note 5.1. Because the differential of a smooth map is supposed to represent the "best linear approximation" to the map near a given point, we can learn a great deal about a map by studying linear-algebraic properties of its differential. The most essential property of the differential—in fact, just about the only property that can be defined independently of choices of bases — is its *rank* (the dimension of the image).

5.1 Maps of constant rank

Note 5.2. We first recall some properties of the rank of a linear map.

Theorem 5.3 (Rank-Nullity [Lee12, Corollary B.21.])

Suppose $T : V \to W$ is a linear map between finite-dimensional vector spaces. Then

$$\begin{aligned} \dim(V) &= \operatorname{rank}(T) + \operatorname{nullity}(T) \\ &= \dim(\operatorname{Im}(T)) + \dim(\ker(T)). \end{aligned}$$

Proposition 5.4 ([Lee12, Exercise B.22.])**.** Suppose V and W are finite dimensional vector spaces and let $T : V \to W$ be a linear map, we have the following.

1. $\operatorname{rank}(T) \leq \dim(V)$, with equality if and only if T is injective.
2. $\operatorname{rank}(T) \leq \dim(W)$, with equality if and only if T is surjective.
3. If $\dim(V) = \dim(W)$ and T is either injective or surjective, then it is an isomorphism.

Definition 5.5. The **rank** of an $n \times m$ matrix A is defined to be rank of the corresponding linear map from $\mathbb{R}^n \to \mathbb{R}^m$. Because the rank of A can also be thought as the dimension of the span of its columns, and is sometimes called it **column rank**. Analogously, we define the **row rank** of A to be the dimension of the span of its rows, thought similarly as vectors in $\mathbb{R}^n$.

Proposition 5.6 ([Lee12, Proposition B.24.])**.** The row rank of a matrix is equal to its column rank.

Definition 5.7. Suppose M and N are smooth manifolds. Given a smooth map $F : M \to N$ and a point $p \in M$, we define the **rank of F at p** to be the rank of the linear map $dF_p : T_pM \to T_{F(p)}N$; it is the rank of the Jacobian matrix of F in any smooth chart, or the dimension of $\operatorname{Im}(dF_p) \subseteq T_{F(p)}N$.

Definition 5.8. If F as above, has the same rank r at every point, we say that it has **constant rank**.

> **Remark 5.9.** Because the rank of a linear map is never higher than the dimension of its domain, the rank of F at each point is bounded above by $\min\{\dim(M), \dim(N)\}$.

Definition 5.10. If the rank of dF_p is equal to this upper bound, we say that F has **full rank at** p, and if F has full rank everywhere, we say F has **full rank**.

5.2 Immersions and submersions

> **Note 5.11.** The most important constant-rank maps are those of full rank.

Theorem 5.12 (Characterisation of injective maps)

Let $T : V \to W$ be a linear map between vector space V and W. We have that T is injective if and only if $\ker(T) = \{0\}$.

> **Note 5.13.** We will use this later.

Definition 5.14. A smooth map $F : M \to N$:

- is called a (smooth) **submersion** if its differential is surjective at each point (or equivalently, if $\operatorname{rank}(F) = \dim(N)$);
- is called a (smooth) **immersion** if its differential is injective at each point (or equivalently, if $\operatorname{rank}(F) = \dim(M)$).

Example 5.15

For a linear map between vector spaces $T : V \to W$ where $\dim(V) = n$ and $\dim(W) = m$,

Property	**Relation between** $\dim V$ and $\dim W$
Injective	$n \leq m$
Surjective	$n \geq m$
Isomorphism (bijective)	$n = m$

Example 5.16

Examples and non-examples.

- Fix $n > m$, the map $\mathbb{R}^n \to \mathbb{R}^m$ such that $(x_1, \ldots, x_m) \mapsto (x_1, \ldots, x_m, 0, \ldots, 0)$ is an immersion.
- Fix $n > m$, the map $\mathbb{R}^n \to \mathbb{R}^m$ such that $(x_1, \ldots, x_m, x_{m+1}, \ldots, x_n) \mapsto (x_1, \ldots, x_m)$ is a submersion.
- A diffeomorphism is both an immersion and a submersion.
- The map $F : \mathbb{R} \to \mathbb{R}^2$ such that $F(t) = (t^2, t^3)$ is not a submersion, since the dimension of the domain is greater than the dimension of the image hence, this map cannot be surjective. The differential

$$dF_t\left(\frac{\partial}{\partial t}\right) = 2t\frac{\partial}{\partial x} + 3t^2\frac{\partial}{\partial y}$$

  is not injective, since $dF_t = 0$ when $t = 0$. Thus this map is not an immersion.
- The map $F : \mathbb{R} \to \mathbb{R}^2$ such that $F(t) = (t^3 - 4t, t^2 - 4)$. It cannot be a submersion due to the reasons outlined above. The differential

$$dF_t\left(\frac{\partial}{\partial t}\right) = (3t^2 - 4)\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial y}$$

  is injective, since it has trivial kernel there are no values of t such that $2t = 0$ and $3t^2 - 4 = 0$ simultaneously. However, note that F itself is not injective as $F(2) = (0, 0) = F(-2)$.

Definition 5.17. If M and N are smooth manifolds, a smooth map $F : M \to N$ is called a **local diffeomorphism** if every point $p \in M$ has a neighbourhood U such that

- $F(U)$ is open in N, and
- $F|_U : U \to F(U)$ is a diffeomorphism.

Example 5.18. Let $M = \mathbb{R} \times \{1\} \cup \mathbb{R} \times \{-1\}$ and $N = \mathbb{R}$ the projection map is a local diffeomorphism.

Theorem 5.19. Suppose $U, V \subset \mathbb{R}^n$ are open subsets, and $F : U \to V$ be a smooth function. If the Jacobian matrix of F is invertible at some point $p \in U$, then there exists an open subset $U_0 \subset U$ containing p such that $F|_{U_0} U_0 \to F(U_0)$ is a diffeomorphism.

Corollary 5.20 (Inverse function theorem for manifolds)
Suppose M and N are smooth manifolds of the same dimension and let $F : M \to N$ be a smooth map. If dF_p is an invertible linear is invertible (as a matrix) then there exists an open neighbourhood U of p such that $F|_U : U \to F(U)$ is a diffeomorphism.

Corollary 5.21. Let M, N be two smooth manifolds of the same dimension. A bijective submersion $F : M \to N$ is a diffeomorphism.

Lemma 5.22. Let $U \subset \mathbb{R}^r$ be an open subset containing the origin. Let $f : U \to \mathbb{R}^n$ be a smooth map of constant rank r (i.e. is an immersion) with $f(0) = 0$. Then there exists an open subset $V \subset \mathbb{R}^n$ containing the origin and a diffeomorphism $g : V \to g(V)$ such that

$$g \circ f(x_1, \ldots, x_r) = (x_1, \ldots, x_r, 0, \ldots, _0).$$

Proof.

To do the proof

□

Theorem 5.23 (Rank theorem)
Let M and N be smooth manifolds of dimensions m and n represent, and $F : M \to N$ is a smooth map with constant rank r. Then for each $p \in M$ there exists smooth chart (U, ϕ) containing p and a smooth chart (V, ψ) containing $F(p)$ such that $F(U) \subset V$ and

$$\widehat{F} = \psi \circ F \circ \phi^{-1} : \mathbb{R}^m \to \mathbb{R}^n$$
$$F(x_1, \ldots, x_r, x_{r+1}, \ldots, x_m) = (x_1, \ldots, x_r, 0, \ldots, 0)$$

5.3 Embeddings

Definition 5.24. Let X and Y be topological spaces. A map $F : X \to Y$ is called a **topological embedding** if F is a homeomorphism $X \to F(X)$, where $F(X)$ is equipped with the subspace topology from Y.

Example 5.25. Consider the set

$$N = \{0\} \cup \{1\} \cup \left\{\frac{1}{2}\right\} \cup \cdots \cup \left\{\frac{1}{N}\right\} \cup \cdots$$

equipped with the discrete topology (i.e. let every subset of N be open (and hence also closed)). The inclusion map $\iota : N \to \mathbb{R}$ is not a topological embedding.

Definition 5.26. Let M and N be two smooth manifolds. A map $F : M \to N$ is called a **smooth embedding** if F is both a topological embedding and an immersion.

Example 5.27

Soem examples.

- The map $\mathbb{R}^n \to \mathbb{R}^{n+k}$ defined by $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, 0 \ldots, k)$ is a smooth embedding.
- Let M and N be smooth manifolds. The inclusion map $\iota_p : M \to M \times N$ where $x \mapsto (x, p)$ for every point $p \in N$ is a smooth embedding.
- Consider $\gamma : \mathbb{R} \to \mathbb{R}^2$ where $\gamma(t) = (t^2, t^3)$. This is clearly a topological embedding. However, notice the differential $\gamma'(t) = 2t\dfrac{\partial}{\partial x} + 3t^2\dfrac{\partial}{\partial y}$ is not injective thus, not an immersion. Therefore, it cannot be a smooth embedding.
- Consider $\gamma : (-\pi, \pi) \to \mathbb{R}^2$ where $\gamma(t) = (\sin 2t, \sin t)$. The differential $\gamma'(t) = 2\cos(2t)\dfrac{\partial}{\partial x} + \cos(t)\dfrac{\partial}{\partial y}$ has trivial kernel, so is a smooth immersion. However, it is not a topological embedding because of problematic behaviour at the origin (this curve looks like a vertical figure 8).

Theorem 5.28 (Local embedding theorem)

Suppose M and N are smooth manifolds and $F : M \to N$ is asmooth map. Then F is a smooth immersion if adn only if for every point $p \in M$ there exists a neighbourhood $U \subset M$ such that $F|_U : U \to N$ is a smooth embedding.

Proof. We prove each direction in turn.

- Proof of ($\Rightarrow$). Suppose F is an immersion, then dF has constant rank at every point $p \in M$. By the constant rank theorem, there exists a neighbourhood of p and charts containing p and $F(p)$ respectively. Say these charts are (U, ϕ) and (V, ψ). Then on U we have that $\psi \circ F \circ \phi^{-1}(x_1, \ldots, x_k) = (x_1, \ldots, x_k, x_{k+1} = 0, \ldots, x_n = 0)$.
- Proof of ($\Leftarrow$). If every point $p \in M$ has a neighbourhood $U \subset M$ such that $F|_U : U \to N$ is a smooth embedding then the derivative of F is an injection at every point, hence F is an immersion.

$\square$

5.4 Submanifolds

Definition 5.29. Suppose M is a smooth manifold, an **embedded submanifold** of M is a subset $S \subseteq M$ if

- it is a manifold in the subspace topology, and
- admits a smooth structure such that the inlcusion $\iota : S \hookrightarrow M$ is a smooth embedding.

Definition 5.30. If S is an embedded manifold of M, the difference $\dim(M) - \dim(S)$ is called the **codimension of** S in M.

Proposition 5.31. Let M be a smooth manifold. The embedded submanifolds of codimension 0 in M are exactly the open subset of M with the induced smooth structure (i.e. open submanifolds).

Proof. If $U \subset M$ is open, then it inherits a smooth manifold structure which makes the inclusion map a smooth embedding. If we have codimension 0 emebedded submanifold S of M, then the inclusion map is a smooth embedding. Particularly, the inclusion map is a local diffeomorphism by the inverse function theroem. Thsi shows that S is an open subset of M. $\square$

Example 5.32. Let $M = \mathbb{R}$ and let $S = (\mathbb{R}, \mathrm{id})$ be its smooth chart. The subset $S = \mathbb{R}$ is a smooth embedding.

Definition 5.33. A continuous map $f : X \to Y$ between topological spaces is called **proper** if for every compact subset $K \subseteq Y$, the preimage $f^{-1}(K) \subseteq X$ is compact.

Definition 5.34. A **properly embedded submanifold** is a smooth embedded submanifold such that the inclusion map is proper.

Example 5.35

An example and non-example.

- If S is a compact smooth embedded submanifold then S is properly embedded.
- Let $S = \{(x, y) : x > 0, y = 0\} \subset \mathbb{R}^2$. Then S is not a proper embedding since the pre-image of the unit open ball in $\mathbb{R}^2$ is an interval of the form $(\cdot, \cdot]$ which is not compact, whereas the open ball is.

5.4.1 Slice charts for embedded submanifolds

Definition 5.36. If $U \subset \mathbb{R}^n$ is open and $k \in \{0, \ldots, n\}$, a **k-dimensional slice of** U (or k-slice) is any subect of the form

$$S = \{(x_1, \ldots, x_k, x_{k+1}, \ldots, x_n) \in U : x_{k+1} = c_{k+1}, \ldots, x_n = c_n\}$$

for some constants $c_{k+1}, \ldots, c_n$.

Theorem 5.37 (Local slice criterion for embedded submanifolds)

Let M be a smooth n-manifold. If $S \subseteq M$ is an embedded k-submanifold . Then for any $p \in S$ there exists a chart (U, ϕ) for M containing p such that $\phi(U \cap S)$ is a k-slice.

Conversely, if S is a subset of M and for any $p \in S$ there exists a chart (U, ϕ) for M containing p such that $\phi(U \cap S)$ is a k-slice, then S, with respect to the subspace topology, admits a smooth structure which makes S an embedded submanifold of dimension k.

Proof. Refer to [Lee12, Theorem 5.8.] $\square$

5.4.2 Level sets

Definition 5.38. If $\phi : M \to N$ is an map and $c \in N$, we call the set $\phi^{-1}(c)$ a **level set** of ϕ.

Theorem 5.39 (Constant rank level set theorem)
Let M and N be smooth manifolds, and let $F : M \to N$ be a smooth map with constant rank r. Each level set of F is a properly embedded submanifold of codimension r in M.

Proof. By the constant rank theorem, for any $q \in F^{-1}(p)$ one can find local charts around q and p such that the level set $F^{-1}(p)$ is a slice with respect to these charts. Hence, it is an embedded submanifold. To show properness, note that $F^{-1}(p)$ is a closed subset of M by continuity. The intersection of a compact set with a closed set is compact. □

Definition 5.40. If $F : M \to N$ is a smooth map, a point $p \in M$ is said to be a **regular point** of F if $dF_p : T_pM \to T_{F(p)}N$ is surjective. Otherwise, it is a **cirtical point** of F.

Definition 5.41. If $F : M \to N$ is a smooth map, a point $q \in N$ is said to be a **regular value** of F if every point of the level set $F^{-1}(q)$ is a regular point. Otherwise, it is a **cirtical value** of F.

Corollary 5.42 (Regular level set theorem)
Let $F : M \to N$ be a smooth map. If $q \in N$ is a regular value of F, then the level set, $F^{-1}(q)$, is a properly embedded submanifold of M. Moreover, we have $\dim(F^{-1}(q)) = \dim M - \dim N$.

Proof.
to do

□

5.5 The tangent space to a submanifold

Note 5.43. If S is a smooth submanifold of $\mathbb{R}^n$, we intuitively think of the tangent space T_pS at a point of S as a subspace of the tangent space $T_p\mathbb{R}^n$. Similarly, the tangent space to a smooth submanifold of an abstract smooth manifold can be viewed as a subspace of the tangent space to the ambient manifold, once we make appropriate identifications.

Let M be a smooth manifold and let S be an embedded submanifold. Since the inclusion map $\iota : S \hookrightarrow M$ is a smooth immersion, at each point $p \in S$ we have an injective linear map $d\iota_p : T_pS \to T_pM$. Recall,

$$d\iota_p(v)(f) = v(f \circ \iota) = v(f|_S).$$

From now on we identify T_pS with its image under this map, thereby thinking T_pS as a linear subspace of T_pM.

Lemma 5.44. Let $f \in C^\infty(M)$ and let S be an emebedded submanifold of M. If $f|_S$ is constant then $v(f) = 0$ for any $v \in T_pS$.

Proposition 5.45

Let $F : M \to N$ be a smooth map and $q \in N$ is a regular value of F. Let S be the submanifold given by $F^{-1}(q)$. For any $p \in S$, we have

$$T_pS = \ker(dF_p).$$

Proof.

TO do

□

Example 5.46

Let $S = \{(x, y) : x^2 + y^2 = 1\} \subset \mathbb{R}^2$ be the unit sphere. It is the level set $F^{-1}(1)$ where $F : \mathbb{R}^2 \to \mathbb{R}$ given by $F(x, y) = x^2 + y^2$. Then at any point $p = (x_0, y_0) \in S$, we have

$$T_pS = \ker(dF_p) = \left\{ a \cdot \left(-y_0 \frac{\partial}{\partial x} + x_0 \frac{\partial}{\partial y} \right) : a \in \mathbb{R} \right\} \subset T_p\mathbb{R}^2.$$

Idk if there is more here?

5.6 The normal bundle

Definition 5.47. Suppose $M \subseteq \mathbb{R}^n$ is an embedded m-dimensional submanifold. For each $x \in M$, we define the **normal space to M at x** to be the $(n - m)$-dimensional subspace $N_xM \subseteq T_x\mathbb{R}^n$ consisting of all the vectors that are orthogonal to T_xM with respect to the Euclidean dot product.

Definition 5.48. The **normal bundle of M**, denoted by NM, is the subset of $T\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n$ consisting of vectors that are normal to M:

$$NM = \{(x, v) \in \mathbb{R}^n \times \mathbb{R}^n : x \in M, v \in N_xM\}.$$

There is a natural projection $\pi_{NM} : NM \to M$ defined as the restriction to NM of $\pi : T\mathbb{R}^n \to \mathbb{R}^n$.

Theorem 5.49

If $M \subseteq \mathbb{R}^n$ is an embedded m-dimensional submanifold, then NM is an embedded n-dimensional submanifold of $T\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n$.

Proof. Let x_0 be any point of M, and let (U, φ) be a slice chart for M in $\mathbb{R}^n$ centered at x_0. Write $\widehat{U} = \varphi(U) \subseteq \mathbb{R}^n$, and write the coordinate functions of φ as $(u^1, \dots, u^n)$, so that $M \cap U$ is the set where $u^{m+1} = \dots = u^n = 0$.

At each point $x \in U$, the vectors

$$E_j\big|_x = (d\varphi_x)^{-1} \left(\frac{\partial}{\partial u^j}\bigg|_{\varphi(x)} \right)$$

form a basis for $T_x\mathbb{R}^n$. We can write

$$E_j\big|_x = E_j^i(x)\frac{\partial}{\partial x^i}\Big|_x,$$

where $E_j^i(x)$ is a smooth function of x (being a partial derivative of φ^{-1}).

Now define a smooth map

$$\Phi : U \times \mathbb{R}^n \to U \times \mathbb{R}^n$$

by

$$\Phi(x,v) = \left(u^1(x), \dots, u^n(x), v \cdot E_1\big|_x, \dots, v \cdot E_n\big|_x\right).$$

The total derivative of Φ at (x,v) is given by the block matrix

$$D\Phi_{(x,v)} = \begin{pmatrix} \frac{\partial u^i}{\partial x^j}(x) & 0 \\ * & E_j^i(x) \end{pmatrix},$$

which is invertible. So Φ is a local diffeomorphism.

If $\Phi(x,v) = \Phi(x',v')$, then $x = x'$ (since φ is injective). The equality of the $v \cdot E_i$ terms implies that $v - v'$ is orthogonal to each $E_i\big|_x$, and hence $v = v'$. Thus Φ is injective and gives a coordinate chart on $U \times \mathbb{R}^n$.

Moreover, $(x,v) \in NM$ if and only if $\Phi(x,v)$ lies in the slice

$$\left\{(y,z) \in \mathbb{R}^n \times \mathbb{R}^n : y^{m+1} = \cdots = y^n = 0,\ z^1 = \cdots = z^m = 0\right\}.$$

Hence, Φ is a slice chart for NM in $\mathbb{R}^n \times \mathbb{R}^n$. □

6 Vector bundles

Note 6.1. We saw that the tangent bundle of a smooth manifold has a natural structure as a smooth manifold in its own right. The natural coordinates we constructed on TM make it look, locally, like the Cartesian product of an open subset of M with $\mathbb{R}^n$. A collection of vector spaces, one for each point in M, glued together in a way that looks *locally* like the Cartesian product of M with $\mathbb{R}^n$, but globally may be "twisted". Such structures are called *vector bundles.*

Definition 6.2. Let M be a smooth manifold. A **(real) smooth vector bundle of rank k over M** is a smooth manifold E together with a surjective smooth map $\pi : E \to M$ satisfying the following conditions:

1. For each $p \in M$, the fibre $E_p = \pi^{-1}(p)$ over p is a k-dimensional real vector space.
2. For each $p \in M$, there exists a neighbourhood U of p in M and a diffeomorphism $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ (called a **(smooth) local trivialisation of E over U**), satisfying the following conditions
   - $\pi_U \circ \Phi = \pi$ (where $\pi_U : U \times \mathbb{R}^k \to U$ is the projection);
   - for each $q \in U$, the restriction $\Phi|_{E_q} : E_q \to \{q\} \times \mathbb{R}^k \cong \mathbb{R}^k$ is a vector space isomorphism.

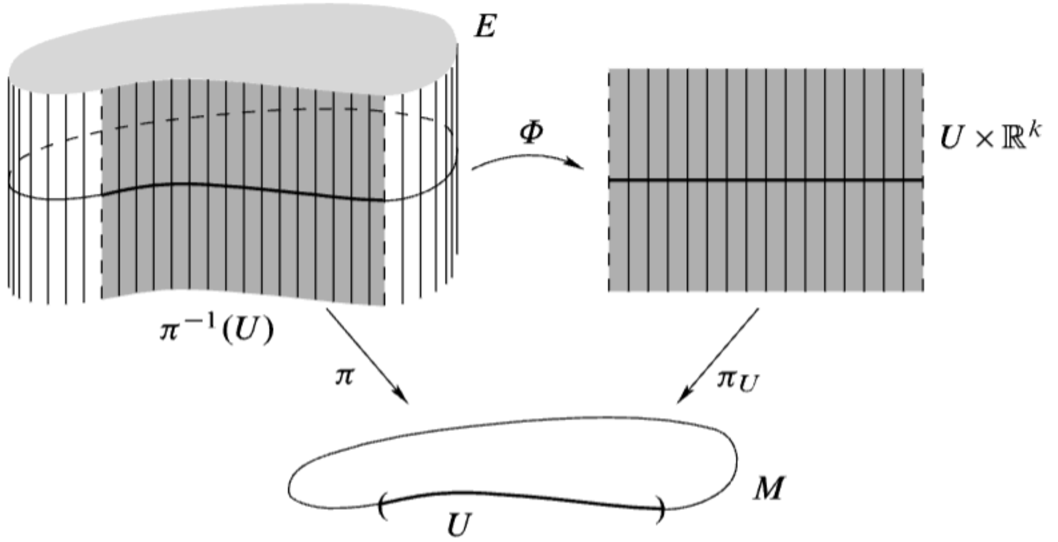

Figure 11: A local trivialisation of a vector bundle [Lee12, Fig. 10.1]

Note 6.3. In American English, 'fibre' is 'fiber'.

Definition 6.4. If there exists a local trivialisation of E over all of M (called a **global trivialisation of** E), then E is said to be a **trivial bundle**.

Example 6.5. For any smooth manifold M, consider the product $M \times \mathbb{R}^k$ for any positive integer k, it is a vector bundle of rank k over M, called the **product bundle**. Note that we can take U to be the whole of M, hence the product bundle admits a global trivialisation.

Proposition 6.6 (The tangent bundle as a vector bundle)
Let M be a smooth n-manifold and let TM be its tangent bundle. With its standard projection, its natural vector space structure on each fibre, and the projection map, TM is a smooth vector bundle of rank n over M.

Proof. Given any smooth chart (U, φ) for M with coordinate functions (x_i), define a map $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^n$ by

$$\Phi\left(v^i \frac{\partial}{\partial x_i}\Big|_p\right) = (p, (v_1, \ldots, v_n)). \tag{10.1}$$

This is linear on fibers and satisfies $\pi_1 \circ \Phi = \pi$. The composite map

$$\pi^{-1}(U) \xrightarrow{\Phi} U \times \mathbb{R}^n \xrightarrow{\varphi \times \mathrm{Id}_{\mathbb{R}^n}} \varphi(U) \times \mathbb{R}^n$$

is equal to the coordinate map $\widetilde{\varphi}$ constructed previously. Since both $\widetilde{\varphi}$ and $\varphi \times \mathrm{Id}_{\mathbb{R}^n}$ are diffeomorphisms, so is Φ. Thus, Φ satisfies all the conditions for a smooth local trivialisation. □ □

Lemma 6.7. Let $\pi : E \to M$ be a smooth vector bundle of rank k over M. Suppose $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ and $\Psi : \pi^{-1}(V) \to V \times \mathbb{R}^k$ are two smooth local trivialisations of E with $U \cap V \neq \emptyset$. There exists a smooth map $\tau : U \cap V \to \mathrm{GL}(k, \mathbb{R})$ such that the composition $\Phi \circ \Psi^{-1} : (U \cap V) \times \mathbb{R}^k \to (U \cap V) \times \mathbb{R}^k$ has the form

$$\Phi \circ \Psi^{-1}(p, v) = (p, \tau(p)v),$$

where $\tau(p)v$ denotes the usual action of the $k \times k$ matrix $\tau(p)$ on the vector $v \in \mathbb{R}^k$.

Proof. The following diagram commutes:

$$\begin{tikzcd}
(U \cap V) \times \mathbb{R}^k \arrow[dr, "\pi_1"'] & \pi^{-1}(U \cap V) \arrow[l, "\Psi"'] \arrow[r, "\Phi"] \arrow[d, "\pi"'] & (U \cap V) \times \mathbb{R}^k \arrow[dl, "\pi_1"] \\
& U \cap V &
\end{tikzcd}$$

where the maps on top are to be interpreted as the restrictions of Ψ and Φ to $\pi^{-1}(U \cap V)$. It follows that $\pi_1 \circ (\Phi \circ \Psi^{-1}) = \pi_1$, which means that

$$\Phi \circ \Psi^{-1}(p, v) = (p, \sigma(p, v))$$

for some smooth map $\sigma : (U \cap V) \times \mathbb{R}^k \to \mathbb{R}^k$. Moreover, for each fixed $p \in U \cap V$, the map $v \mapsto \sigma(p, v)$ from $\mathbb{R}^k$ to itself is an invertible linear map, so there is a nonsingular $k \times k$ matrix $\tau(p)$ such that $\sigma(p, v) = \tau(p)v$. It remains only to show that the map $\tau : U \cap V \to \mathrm{GL}(k, \mathbb{R})$ is smooth. This is left to Problem **10.4**. $\square$

Lemma 6.8 (Vector Bundle Chart Lemma)

Let M be a smooth manifold with or without boundary, and suppose that for each $p \in M$ we are given a real vector space E_p of some fixed dimension k. Let $E = \bigsqcup_{p \in M} E_p$, and let $\pi : E \to M$ be the map that takes each element of E_p to the point p. Suppose furthermore that we are given the following data:

(i) an open cover $\{U_\alpha\}_{\alpha \in A}$ of M,

(ii) for each $\alpha \in A$, a bijective map $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$ whose restriction to each E_p is a vector space isomorphism from E_p to $\{p\} \times \mathbb{R}^k \cong \mathbb{R}^k$,

(iii) for each $\alpha, \beta \in A$ with $U_\alpha \cap U_\beta \neq \emptyset$, a smooth map $\tau_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(k, \mathbb{R})$ such that the map $\Phi_\alpha \circ \Phi_\beta^{-1}$ from $(U_\alpha \cap U_\beta) \times \mathbb{R}^k$ to itself has the form

$$\Phi_\alpha \circ \Phi_\beta^{-1}(p, v) = (p, \tau_{\alpha\beta}(p)v). \qquad (10.3)$$

Then E has a unique topology and smooth structure making it into a smooth manifold with or without boundary and a smooth rank-k vector bundle over M, with π as projection and $\{(U_\alpha, \Phi_\alpha)\}$ as smooth local trivializations.

6.1 Section of vector bundles

Definition 6.9. Let $\pi : E \to M$ be a vector bundle. A **section of** E (someitmes called a **cross section**) is a section of the map π, that is, a smooth map

$$\sigma : M \to E \quad \text{such that} \quad \pi \circ \sigma = \mathrm{id}_M .$$

Definition 6.10. (More generally), a **local section of** E is a smooth map $\sigma : U \to E$ defined on some open subset $U \subseteq M$ and satisfying $\pi \circ \sigma = \mathrm{id}_U$.

Example 6.11

For any smoth vector bundle $\pi : E \to M$, the **zero section of** E $\sigma : M \to E$ is defined by

$$\sigma(p) = 0 \in E_p$$

for each $p \in M$.

6.2 Frames

Definition 6.12. Let $E \to M$ be a vector bundle. A collection of smooth sections $\sigma_1, \ldots, \sigma_i$ is called **linearly independent** if $\sigma_1(p), \ldots, \sigma_i(p)$ are linearly independent vectors in E_p for any $p \in M$.

Definition 6.13. For a rank-k smooth vector bundle, a **frame** is a collection of k linearly independent sections.

Example 6.14

For a trivial vector bundle $E = X \times \mathbb{R}^k$, let $e_1, \cdots, e_k$ be a basis of $\mathbb{R}^k$. Then

$$\{\sigma_i : X \to E, \quad \sigma_i(p) := (p, e_i)\}_{1 \leq i \leq k}$$

form a frame of E. The converse of this is also true.

Proposition 6.15

If a vector bundle $\pi : E \to M$ admits a frame, then it admits a global trivialisation $\Psi : M \times \mathbb{R}^k \to E$.

Proof. Let $\sigma_1, \cdots, \sigma_k$ be a frame of E. Then for any $v \in E_p$ we have $v = \sum_i a_i(p)\sigma_i(p)$. Set

$$\Psi : X \times \mathbb{R}^k \to E, \quad \Psi(p, a_1, \cdots, a_k) := \left(p, \sum_i a_i \sigma_i(p)\right).$$

We will show Ψ is a diffeomorphism. Note that Ψ is bijective, we only need to show it is a local diffeomorphism.

Pick a local trivialization $(U, \Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k)$. So we can compose the two maps

$$U \times \mathbb{R}^k \xrightarrow{\Psi} \pi^{-1}(U) \xrightarrow{\Phi} U \times \mathbb{R}^k.$$

For each σ_i, $\Phi \circ \sigma_i$ is a smooth map. Thus there exist smooth functions $\sigma_i^1, \cdots, \sigma_i^k : U \to \mathbb{R}$ such that

$$\Phi(p, \sigma_i(p)) = (p, \sigma_i^1(p), \cdots, \sigma_i^k(p)).$$

Therefore the composition $\Phi \circ \Psi$ has the form

$$(p, a_1, \cdots, a_k) \mapsto \left(p, \left(\sum_i a_i \sigma_i^1(p)\right), \cdots, \sum_i a_i \sigma_i^k(p)\right).$$

It is smooth since σ_i^j's are smooth. Hence $\Phi \circ \Psi$ is smooth. Similarly, by considering maps in the opposite direction we can show $(\Phi \circ \Psi)^{-1}$ is smooth. Therefore Ψ is a local diffeomorphism. $\square$

Corollary 6.16

If a smooth k-manifold M admits a k smooth vector field which are linearly independent at avery point, then the tangent bundle of M is trivial.

Example 6.17. Consider the unit circle $S = \{x^2 + y^2 = 1\} \subset \mathbb{R}^2$ and a smooth map $F : \mathbb{R} \to S$, $F(t) = (\cos(t), \sin(t))$. It is an immersion, hence the nowhere vanishing vector field $(t, \frac{\partial}{\partial t})$ on $\mathbb{R}$ is pushed to a nowhere vanishing vector field on S. Therefore S has a trivial tangent bundle.

6.3 The cotangent bundle

6.4 Covectors

Definition 6.18. Let V be a finite-dimensional vector space over $\mathbb{R}$. A **covector on** V is a real-valued linear functional on V, that is, a linear map $\omega : V \to \mathbb{R}$.

Definition 6.19. THe sapce of all covectors on V is itself a real vector space under the obvious operations of pointwise addition and scalar multiplication. It is denoted by V^* and called the **dual space of** V.

Proposition 6.20

Let V be a finite-dimensional vector space. Given any basis $\{E_1, \ldots, E_n\}$ for V, let $\varepsilon_1, \ldots, \varepsilon_n \in V^*$ be the covectors defined by

$$\varepsilon_i(E_j) = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j. \end{cases}$$

Then $\{\varepsilon_1, \ldots, \varepsilon_n\}$ is a basis for V^*, called the **dual basis to** $\{E_j\}$. Therefore, $\dim V^* = \dim V$.

6.5 Tangent covectors on manifolds

Definition 6.21. Let M be a smooth manifold. For each $p \in M$, we define the **cotangent space at** p, denoted by T_p^*M, to be the dual space to T_pM:

$$T_p^*M = (T_pM)^*.$$

Elements of T_p^*M are called **tangent covectors at** p, or just **covectors at** p.

Example 6.22 (Covector basis)
Let (U, φ) be a smooth chart on M with coordinate functions $x_1, \ldots, x_n$. Recall for each $p \in U$ we have a basis

$$\left\{ \frac{\partial}{\partial x_i}\bigg|_p : i \in 1, \ldots, n \right\}$$

for T_pM. Let $\lambda_1|_p, \ldots, \lambda_n|_p$ be the dual basis. Then any covector $\omega \in T_p^*M$ can be written as

$$\omega = \sum_i \omega_i \lambda_i|_p \quad \text{where} \quad w_i = w\left(\frac{\partial}{\partial x_i}\bigg|_p \right) \in \mathbb{R}.$$

Therefore the chart (U, φ) give a coordinate expression for covectors.

Example 6.23 (Coordinate change of basis)
Suppose (V, ψ) is another chart containing p, with coordinate functions $\tilde{x}_1, \cdots, \tilde{x}_n$. The coordinate vector transform as follows

$$\frac{\partial}{\partial x_i}\bigg|_p = \sum_j \frac{\partial \tilde{x}_j}{\partial x_i}(p) \frac{\partial}{\partial \tilde{x}_j}\bigg|_p.$$

(Here we use p to denote either a point in X or its coordinate representation as appropriate.) Writing ω in both coordinates as

$$\omega = \sum_i \omega_i \lambda_i|_p = \sum_j \widetilde{\omega}_j \tilde{\lambda}_j|_p,$$

the components ω_i transform as follows

$$\omega_i = \sum_j \frac{\partial \tilde{x}_j}{\partial x_i}(p) \widetilde{\omega}_j.$$

By using this coordinate change formula, we can verify that T^*X is a smooth manifold.

Proposition 6.24
Let M be a smooth n-manifold. Its cotangent bundle T^*M admits a smooth manifold structure with dimension $2n$.

Proof. Let (U, ϕ) be a chart on X and let $\pi : T^*X \to X$ be the projection. The map

$$\tilde{\phi} : \pi^{-1}(U) \to \mathbb{R}^{2n}, \quad \tilde{\phi}(p, \omega) = (\phi(p), \omega_1, \cdots, \omega_n)$$

is a homeomorphism. The transition maps between different charts are computed as above, which are smooth. □

Definition 6.25. Similar to vector fields, we can define the notion of **covector field** as a smooth map

$$\sigma : X \to T^*X, \quad \pi \circ \sigma = \mathrm{Id}.$$

Proposition 6.26
In other words, a covector field is a smooth section of the cotangent bundle. Given any smooth function $f : X \to \mathbb{R}$, it gives a covector field df by

$$(df)_p(v) := v(f), \quad \forall p \in X, v \in T_pX.$$

Example 6.27
Let (U, ϕ) be a chart with coordinate functions $x_1, \cdots, x_n$. Let $\lambda_1, \cdots, \lambda_n$ be the dual basis to $\partial/\partial x_1, \cdots, \partial/\partial x_n$. Then one can directly compute that

$$(df)_p = \sum_i \frac{\partial f}{\partial x_i}(p)\lambda_i\big|_p.$$

In particular, we get $(dx_i)_p = \lambda_i\big|_p$ for any i. From now on, we will also use $dx_1, \cdots, dx_n$ as the dual basis to $\dfrac{\partial}{\partial x_1}, \cdots, \dfrac{\partial}{\partial x_n}$, once a chart is chosen.
Therefore, the formula above can be written as

$$df_p = \sum_i \frac{\partial f}{\partial x_i} dx_i|_p.$$

In particular in the 1-dimensional case, this reduces to

$$df = \frac{\mathrm{d}f}{\mathrm{d}x} dx.$$

Example 6.28
If $f(x, y) = x^2 y \cos x$ on $\mathbb{R}^2$, then df is given by the formula

$$df = \frac{\partial(x^2 y \cos x)}{\partial x} dx + \frac{\partial(x^2 y \cos x)}{\partial y} dy = (2xy \cos x - x^2 y \sin x)\, dx + x^2 \cos x\, dy.$$

6.6 Pullbacks of covector fields

Definition 6.29. Let $\mathfrak{X}^*(M)$ be the space of covector fields on M. Similar to the space of vector fields on M, it is a vector field over $\mathbb{R}$. Moreover, for any smooth function f on M and $\sigma \in \mathfrak{X}^*(M)$, we get a new covector field $f \cdot \sigma \in \mathfrak{X}^*(M)$, by setting

$$(f \cdot \sigma)_p := f(p) \cdot \sigma(p).$$

Definition 6.30. Let $F : M \to N$ be a smooth map between smooth manifolds, and let $p \in M$ be arbitrary. The differential $dF_p : T_pM \to T_{F(p)}N$ yields a dual linear map

$$\begin{aligned} dF_p^* &: T^*_{F(p)}N \to T^*_pM \\ dF_p^*(\omega)(v) &= \omega(dF_p(v)) \end{aligned}$$

for $\omega \in T^*_{F(p)}N$ and $v \in T_pM$, called the **pullback by F at p**, or the **cotangent map of F**.

Proposition 6.31

Let $F : M \to N$ be a smooth map. For any $\omega \in \mathfrak{X}^*(Y)$, define

$$(F^*\omega)_p := dF^*_p(\omega_{F(p)}).$$

Then $F^*\omega$ is a smooth covector field on M. This is called the **pullback of ω by F.**

Proof. The proof follows from computation in coordinates, based on the lemma below. Let $p \in X$ and V be a smooth chart on Y containing $F(p)$. Let $(y_1, \cdots, y_n)$ be the coordinate functions on V. Then we can write ω in coordinates as $\omega = \sum_j \omega_j dy_j$. Since ω is smooth, the functions ω_j's are smooth. Apply the following lemma to the smooth map $F\big|_{F^{-1}(V)} : F^{-1}(V) \to V$, we get

$$F^*\omega = F^*\left(\sum_j \omega_j dy_j\right) = \sum_j (\omega_j \circ F)F^*(dy_j) = \sum_j (\omega_j \circ F)d(y_j \circ F).$$

Since ω_j, y_j, F are smooth, the pull back is smooth. $\square$

Lemma 6.32

Let $F : X \to Y$ be a smooth map.

1. For $\omega, \eta \in \mathfrak{X}^*(Y)$, we have $F^*(\omega + \eta) = F^*\omega + F^*\eta$.
2. For $f \in C^\infty(Y)$ and $\omega \in \mathfrak{X}^*(Y)$, we have $F^*(f\omega) = (f \circ F)F^*\omega$.
3. For $f \in C^\infty(Y)$, $F^*(df) = d(f \circ F)$.

Example 6.33

Let $F : \mathbb{R}^3 \to \mathbb{R}^2$ be the map given by

$$F(x, y, z) = (x^2 y, y \sin z) = (u, v).$$

and let $\omega \in \mathfrak{X}^*(\mathbb{R}^2)$ be the covector field

$$\omega = udv + vdu.$$

Compute $F^*\omega$.

Solution. Using the formula from the above proof:

$$F^*\omega = \sum_j (\omega_j \circ F) d(y_j \circ F).$$

where $\omega = \sum_j \omega_j dy_j$ we have $\omega_1 = u, \omega_2 = v, y_1 = v$ and $y_2 = u$. Therefore, the pullback is given by

$$\begin{aligned} F^*\omega &= (u \circ F)\, d(v \circ F) + (v \circ F)\, d(u \circ F) \\ &= (x^2 y)\, d(y \sin z) + (y \sin z)\, d(x^2 y) \\ &= x^2 y(\sin z\, dy + y \cos z\, dz) + y \sin z(2xy\, dx + x^2\, dy) \\ &= 2xy^2 \sin z\, dx + 2x^2 y \sin z\, dy + x^2 y^2 \cos z\, dz. \end{aligned}$$

Note that u and v are coordinate functions, i.e. $u(x, y) = x$ and $v(x, y) = y$.

7 Differential forms

7.1 Tensors

Note 7.1. Much of the technology of smooth manifold theory is designed to allow the concepts of linear algebra to be applied to smooth manifolds. In this section, we summarise results from multi-linear algebra, taken from [Lee12, Chapter 12].

Definition 7.2. Let V be a finite dimensional vector space over $\mathbb{R}$. If $k > 0$, a **covariant k-tensor on V** is an element of $\underbrace{V^* \otimes \cdots \otimes V^*}_{k \text{ copies}}$, which we think of as a real-valued multilinear function on k elements of V:

$$\alpha : \underbrace{V \times \cdots \times V}_{k \text{ copies}} \to \mathbb{R}.$$

The number k is called the **rank of α**.

Note 7.3. This definition is not necessary for the purposes of this course, but it is useful when reading the source material [Lee12, Chapter 12].

Definition 7.4. We denote the vector space of all covarinat k-tensors on V by the shorthand notaion

$$T^kV^* = \underbrace{V^* \otimes \cdots \otimes V^*}_{k \text{ copies}}$$

Example 7.5

We see some examples:

- When $k = 0$, we have that $T^0V^* = \mathbb{R}$ since its elements are real-valued functions depending multilinearly on no vectors.
- When $k = 1$, we have that $T^1V^* = V^*$ since every linear functional $\omega : V \to \mathbb{R}$ is multilinear.
- For $k > 0$ every element of T^kV^* is a multilinear functional on the product of k copies of V.

Definition 7.6. An element $\omega \in T^kV^*$ is called a degree-k **alternating element**, or just a **k-form**, if for all vector $v_1, \dots, v_k \in V$ and every pair of distinct indices i, j we have

$$\omega(v_1, \dots, v_i, \dots, v_j, \dots, v_k) = -\omega(v_1, \dots, v_j, \dots, v_i, \dots, v_k).$$

That is, when we swap two entries we get a minus sign. The space of all degree-k alternating elenments is denoted by A^kV^*.

Theorem 7.7

Let V be a n-dimensional vector space over $\mathbb{R}$. Then A^kV^* is a vector subspace of T^kV^*. Furthermore, if $k > n$ then $A^kV^* = \varnothing$.

Proposition 7.8 ([Lee12, Exercise 12.17.])**.** Any permutation $\sigma \in S_k$, the **sign of** σ, denoted by

$$\operatorname{sgn}(\sigma) = \begin{cases} +1 & \text{if } \sigma \text{ is even} \\ -1 & \text{if } \sigma \text{ is odd.} \end{cases}$$

For any $\omega \in A^kV^*$, for any vectors $v_1, \dots, v_k$ and any permutation $\sigma \in S_k$

$$\omega(v_{\sigma(1)}, \dots, v_{\sigma(k)}) = (\operatorname{sgn}(\sigma))\omega(v_1, \dots, v_k).$$

7.2 The wedge product

Note 7.9. In this section we define a product operation for alternating tensors.

Definition 7.10. Let V be a finite dimensional vector space over $\mathbb{R}$. Given $\omega \in A^kV^*$ and $\eta \in A^lV^*$, we define their **wedge product**:

$$\omega \wedge \eta : A^kV^* \times A^lV^* \to A^{k+l}V^*$$
$$(\omega \wedge \eta)(v_1, \dots, v_{k+l}) = \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \operatorname{sgn}(\sigma)\omega(v_{\sigma(1)}, \dots, v_{\sigma(k)})\eta(v_{\sigma(k+1)}, \dots, v_{\sigma(k+l)})$$

Example 7.11
Let $\omega \in A^1V^*$ and $\eta \in A^2V^*$. Then we have:

$$\omega \wedge \eta(v_1, v_2, v_3) = \frac{1}{1!2!}\Big(\omega(v_1)\eta(v_2, v_3) - \omega(v_1)\eta(v_3, v_2)$$
$$-\omega(v_2)\eta(v_1, v_3) + \omega(v_2)\eta(v_3, v_1)$$
$$+\omega(v_3)\eta(v_1, v_2) - \omega(v_3)\eta(v_2, v_1)\Big).$$

It is equivalent to:

$$\omega \wedge \eta(v_1, v_2, v_3) = \omega(v_1)\eta(v_2, v_3) - \omega(v_2)\eta(v_1, v_3) + \omega(v_3)\eta(v_1, v_2).$$

Example 7.12 (Tricks)
Steps:

1. Start with the smallest number not yet used.
2. Follow the map $i \mapsto \sigma(i)$ until it loops back.
3. Repeat with the next smallest unused number.

Example (one-line notation): $\sigma = [3\ 5\ 2\ 1\ 4]$
This means:

$$\sigma(1) = 3, \quad \sigma(3) = 2, \quad \sigma(2) = 5, \quad \sigma(5) = 4, \quad \sigma(4) = 1 \Rightarrow \sigma = (1\ 3\ 2\ 5\ 4)$$

The the number of transpositions in a cycle. Each cycle of length k contributes $k-1$ transpositions.
Formula:

$$\#\text{ transpositions} = \sum_{\text{cycles}} (\text{length} - 1)$$

Example:

$$\sigma = (1\ 4\ 5)(2\ 3) \Rightarrow 2 + 1 = 3 \text{ transpositions} \Rightarrow \operatorname{sgn}(\sigma) = (-1)^3 = -1$$

It follows that the sign of the permutation is the given by:

$$\operatorname{sgn}(\sigma) = (-1)^{\#\text{ transpositions in cycle decomposition}}$$

Use the rule above to avoid writing out the full transpositions.

Proposition 7.13. If $\{\omega_1, \ldots, \omega_n\}$ is a basis V^* (i.e. $\dim(V^*) = n$) then

$$\{\omega_{i_1} \wedge \cdots \wedge \omega_{i_k} : i_1 < \ldots < i_p\}$$

is a basis of A^kV^* over $\mathbb{R}$. Hence, $\dim(A^kV^*) = \binom{n}{k} = \frac{n!}{k!(n-k)!}$.

Proposition 7.14 (Properties of the Wedge Product)
Suppose $\omega, \omega', \eta, \eta'$, and ξ are multivectors on a finite-dimensional vector space V.

1. **Bilinearity:** *For* $a, a' \in \mathbb{R}$,
$$(a\omega + a'\omega') \wedge \eta = a(\omega \wedge \eta) + a'(\omega' \wedge \eta),$$
$$\eta \wedge (a\omega + a'\omega') = a(\eta \wedge \omega) + a'(\eta \wedge \omega').$$
2. **Associativity:**
$$\omega \wedge (\eta \wedge \xi) = (\omega \wedge \eta) \wedge \xi.$$
3. **Anticommutativity:** *For* $\omega \in \Lambda^k(V^*)$ *and* $\eta \in \Lambda^l(V^*)$,
$$\omega \wedge \eta = (-1)^{kl}\eta \wedge \omega.$$

7.3 Differential forms on manifolds

Theorem 7.15
Let M be a smooth n-dimensional manifold. For any $k > 0$ the set
$$A^k(T^*M) = \bigsqcup_{p \in M} A^k(T_p^*M)$$
admits a smooth manifold structure, and it is a smooth vector bundle over M. Furthermore, for $k = 0$ we define $A^0(T^*M) = M \times \mathbb{R}$ as the product bundle.

Definition 7.16. Let M be a smooth manifold. A **differential k-form** is a smooth section of the vector bundle $A^k(T^*M)$. The integer k is called the **degree** of the form. We denote the vector space of differential k-forms by $\Omega^k(M)$.

Example 7.17. Examples of $\Omega^k(M)$.

- $\Omega^0(M)$ is the space of smooth functions on M i.e. $\Omega^0(M) = C^\infty(M)$.
- $\Omega^1(M)$ is the space of covector fields on M i.e. $\Omega^1(M) = \mathfrak{X}^*(M)$.

Example 7.18

Let x, y, z be the coordinate functions on $\mathbb{R}^3$.

1. A 0-form is just a continuous real-valued function.
2. A 1-form is a covector field i.e. $\omega = f_1 dx + f_2 dy + f_3 dz$ for some smooth functions f_1, f_2, f_3.
3. A 2-form is $\eta = g_1 dx \wedge dy + g_2 dx \wedge dz + g_3 dy \wedge dz$ for some smooth funcions g_1, g_2, g_3.
4. Every 3-form on $\mathbb{R}^3$ is a continuous real-valued function times $dx \wedge dy \wedge dz$.

, and . On $\mathbb{R}^3$, some examples of smooth 2-forms are given by

Proposition 7.19

In general, let $x_1, \dots, x_n$ be the coordinate funcions on $\mathbb{R}^n$. Then a differential k-form on $\mathbb{R}^n$ can be written as

$$\omega = \sum_{i_1 < \cdots < i_k} f_{i_1, \dots, i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

for smooth functions $f_{i_1, \dots, i_k} \in C^\infty(\mathbb{R}^n)$.

Definition 7.20. If $F : M \to N$ is a smooth map and ω is a differential form on N, the **pullback**

$$F^* : \Omega^k(N) \to \Omega^k(M)$$
$$(F^*\omega)_p(v_1, \dots, v_k) = \omega_{F(p)}(dF_p(v_1), \dots, dF_p(v_k)).$$

If $k = 0$ then we define $F^* f = f \circ F$.

Lemma 7.21 (Properties of pullback)**.** Suppose $F : M \to N$ is smooth.

1. $F^* : \Omega^k(N) \to \Omega^k(M)$ is linear over $\mathbb{R}$.
2. $F^*(\omega \wedge \eta) = (F^*\omega) \wedge (F^*\eta)$.
3. If F is a diffeomorphism $(F^*)^{-1} = (F^{-1})^*$.
4. For $G : N \to Z$ we have $F^* \circ G^* = (G \circ F)^*$.

Proposition 7.22 (Pullback in coordinates)

Given a smooth map

$$F : \mathbb{R}^n \to \mathbb{R}^m$$

The pullback is given by

$$F^* \left(\sum_j \omega_j d(y_1) \wedge \cdots \wedge dy_j \right) = \sum_j (\omega_j \circ F)(dy_1 \circ F) \wedge \cdots \wedge (dy_j \circ F).$$

Example 7.23

Define $F : \mathbb{R}^2 \to \mathbb{R}^3$ by $F(u, v) = (u, v, u^2 - v^2)$, and let ω be the 2-form $y\, dx \wedge dz + x\, dy \wedge dz$ on $\mathbb{R}^3$. The pullback $F^*\omega$ is computed as follows:

$$\begin{aligned} F^*(y\, dx \wedge dz + x\, dy \wedge dz) &= F^*(ydx \wedge dz) + F^*(xdy \wedge dz) \\ &= (y \circ F)d(x \circ F) \wedge d(z \circ F) + (x \circ F)d(y \circ F) \wedge d(z \circ F) \\ &= v\, du \wedge d(u^2 - v^2) + u\, dv \wedge d(u^2 - v^2) \\ &= v\, du \wedge (2u\, du - 2v\, dv) + u\, dv \wedge (2u\, du - 2v\, dv) \\ &= -2v^2\, du \wedge dv + 2u^2\, dv \wedge du, \end{aligned}$$

where we have used the fact $du \wedge du = dv \wedge dv = 0$ by anticommutativity. Because $dv \wedge du = -du \wedge dv$, this simplifies to

$$F^*\omega = -2(u^2 + v^2)\, du \wedge dv.$$

7.4 Exterior derivative

Note 7.24. In this section we define a natural differential operator on smooth forms, which is a generalisation of the differential of a function.

Definition 7.25. For a general k-from

$$\omega = \sum_{i_1 < \cdots < i_k} f_{i_1,\ldots,i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

in $\mathbb{R}^n$ the **exterior derivative** is defined as the map

$$\begin{aligned} d : \Omega^k(\mathbb{R}^n) &\to \Omega^{k+1}(\mathbb{R}^n) \\ d\omega &= \sum_{i_1 < \cdots < i_k} (df_{i_1,\ldots,i_k}) \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_p}. \end{aligned}$$

Example 7.26

For a smooth 0-from f (a smooth function) this definition reduces to

$$df = \sum_i \frac{\partial f}{\partial x_i} dx_i$$

which is just the differential of f.

Proposition 7.27 (Properties of the exterior derivative on $\mathbb{R}^n$)
We list some properties.

1. d is a linear operator over $\mathbb{R}$.
2. $d \circ d = 0$.
3. (**Graded Leibniz rule**) Let $\omega \in \Omega^k(\mathbb{R}^n)$ and $\eta \in \Omega^l(\mathbb{R}^n)$ then
$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta.$$
4. d commutes with pullback. Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a smooth map, we have
$$F^*(dw) = d(F^*\omega)$$
for any differential form ω.

Proof. The first proof is by direct definition of the operator. The remaining proof follow from direct computation. □

Example 7.28 (Exterior Derivatives and Vector Calculus in $\mathbb{R}^3$)
Let us work out the exterior derivatives of arbitrary 1-forms and 2-forms on $\mathbb{R}^3$. Any smooth 1-form can be written

$$\omega = P\,dx + Q\,dy + R\,dz$$

for some smooth functions P, Q, R. Using (14.19) and the fact that the wedge product of any 1-form with itself is zero, we compute

$$\begin{aligned}
d\omega &= dP \wedge dx + dQ \wedge dy + dR \wedge dz \\
&= \left(\frac{\partial P}{\partial x}\,dx + \frac{\partial P}{\partial y}\,dy + \frac{\partial P}{\partial z}\,dz\right) \wedge dx + \left(\frac{\partial Q}{\partial x}\,dx + \frac{\partial Q}{\partial y}\,dy + \frac{\partial Q}{\partial z}\,dz\right) \wedge dy \\
&\quad + \left(\frac{\partial R}{\partial x}\,dx + \frac{\partial R}{\partial y}\,dy + \frac{\partial R}{\partial z}\,dz\right) \wedge dz \\
&= \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy + \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) dx \wedge dz + \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) dy \wedge dz.
\end{aligned}$$

An arbitrary 2-form on $\mathbb{R}^3$ can be written

$$\eta = u\,dx \wedge dy + v\,dx \wedge dz + w\,dy \wedge dz.$$

A similar computation shows that

$$d\eta = \left(\frac{\partial u}{\partial z} - \frac{\partial v}{\partial y} + \frac{\partial w}{\partial x}\right) dx \wedge dy \wedge dz.$$

7.4.1 Exterior derivative on manifolds

Definition 7.29. Suppose $\omega \in \Omega^k(M)$. Pick a chart (U, φ) for M, we define the exterior derivative on M as

$$d\omega = \varphi^* d((\varphi^{-1})^* \omega)$$

Note 7.30. The inside d is the total differential of the map.

Proposition 7.31. The operator $d : \Omega^p(X) \to \Omega^{p+1}(X)$ is well-defined.

Proof. Since our definition involves a choice of chart, we need to show the independence of such a choice. Pick another chart (V, ψ) containing x, we will check

$$(\phi^* d(\phi^{-1})^* \omega)_x = (\psi^* d(\psi^{-1})^* \omega)_x.$$

Note that

$$\begin{aligned}(\phi^{-1})^* \circ \psi^* (d(\psi^{-1})^* \omega) &= (\psi \circ \phi^{-1})^* (d(\phi^{-1})^* \omega) \\ &= d((\psi \circ \phi^{-1})^* \circ (\psi^{-1})^* \omega) \\ &= d((\phi^{-1})^* \circ \psi^* \circ (\psi^{-1})^* \omega) \\ &= d((\phi^{-1})^* \omega).\end{aligned}$$

The first and the third equality are the composition law for the pullback. The second equality is the commutativity law. Now we apply ϕ^* on both sides of the equation we get $(\psi^* d(\psi^{-1})^* \omega) = (\phi^* d(\phi^{-1})^* \omega)$. □

Corollary 7.32. The operator $d : \Omega^p(X) \to \Omega^{p+1}(X)$ satisfies that

1. $d \circ d = 0$.
2. $F^* \circ d = d \circ F^*$ for any smooth map $F : X \to Y$.
3. $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^p \omega \wedge d\eta$ for $\omega \in \Omega^p(X), \eta \in \Omega^q(X)$.

Proof. (1) and (3) follow from computation in coordinates. We prove (2) here. Pick a chart (U, ϕ) on X and a chart (V, ψ) on Y. Then $\psi \circ F \circ \phi^{-1}$ is a smooth map between Euclidean spaces. Pick $\omega \in \Omega^p(Y)$. Using Proposition **6.17** we get

$$(\psi \circ F \circ \phi^{-1})^* d(\psi^{-1})^* \omega = d(\psi \circ F \circ \phi^{-1})^* (\psi^{-1})^* \omega.$$

The composition law for pull back shows the right hand side can be simplified to $d(\phi^{-1})^* F^* \omega$. Therefore we get

$$\begin{aligned}d(F^* \omega) &= \phi^* d(\phi^{-1})^* F^* \omega \\ &= \phi^* (\psi \circ F \circ \phi^{-1})^* d(\psi^{-1})^* \omega \\ &= F^* \psi^* d(\psi^{-1})^* \omega \\ &= F^* (d\omega).\end{aligned}$$

The first and last equality are by definition. The second equality is given above. The third equality is the composition law for pull back.

□

7.5 De Rham cohomology

Note 7.33. This is an invariant of smooth manifolds.

Definition 7.34. We provide two definitions.

- A differential k-form is called **closed** if $d\omega = 0$, and
- is called **exact** if there exists a differential $(k-1)$-form η such that $d\eta = \omega$.

Definition 7.35. Let M be a smooth manifold, and let p be a non-negative integer. Because $d : \Omega^p(M) \to \Omega^{p+1}(M)$ is linear, its kernel and image are linear subspaces. We define

$$Z^p(M) = \ker\left(d : \Omega^p(M) \to \Omega^{p+1}(M)\right) = \{\text{closed } p\text{-forms on } M\},$$
$$B^p(M) = \operatorname{Im}\left(d : \Omega^p(M) \to \Omega^{p+1}(M)\right) = \{\text{exact } p\text{-forms on } M\}$$

By convention, we consider $\Omega^p(M)$ ot be the zero vector space when $p < 0$ or $p > \dim(M)$, so that, for example $B^(M) = 0$ and $Z^{\dim(M)}(M) = \Omega^{\dim(M)}(M)$.

Proposition 7.36. The fact that every exact form is closed implies $B^p(M) \subseteq Z^p(M)$.

Definition 7.37. Let M be a smooth manifold. The **degree p de Rham cohomology group** of M is defined as

$$H^p_{\mathrm{dR}}(M) = \frac{Z^p(M)}{B^p(M)}$$

(a quotient vector space). By definition $H^p_{\mathrm{dR}}(M) = 0$ if $p < 0$ or $p > \dim(M)$.

Proposition 7.38

Let M be a smooth manifold. Then $H^0_{\mathrm{dR}}(M)$ is isomorphic to $\mathbb{R}^k$, where k is the number of connected components of M.

Proof. Since $\Omega^{-1}(M) = 0$, we have $B^0(M) = 0$. Hence we only need to compute $Z^0(M)$. Pick a smooth function f on M with $df = 0$. Pick any point $x \in M$ and consider the subset

$$U := \{p \in M : f(p) = f(x)\}.$$

By definition, the set U is closed. On the other hand, it is open by using local coordinate computation and $df = 0$. Therefore, U is a connected component of M, and $H^0_{\mathrm{dR}}(M)$ is isomorphic to the space of functions that are constant on each component. □

Proposition 7.39

$H^1_{\mathrm{dR}}(\mathbb{R}) = 0$.

Proof. Pick a one-form $f(x)dx$, it is always closed since $\Omega^2(\mathbb{R}) = 0$. On the other hand, set

$$g(x) := \int_0^x f(t)dt.$$

We get $dg = fdx$. Hence any one-form is also exact. □

8 Integration

8.1 Orientation of smooth manifolds

Theorem 8.1

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a diffeomorphism. Let x_i be the coordinates on the domain and y_i be the coordinates on the codomain. Then we have that

$$T^* (dy_1 \wedge \cdots \wedge dy_n) = \det(J_T) dx_1 \wedge \cdots \wedge dx_n$$

where $\det(J_T)$ is the determinant of the Jacobian matrix of T.

Note 8.2. We will use this to define orientation.

Definition 8.3. A diffeomorphism $T : \mathbb{R}^n \to \mathbb{R}^n$ is called **orientation preserving** if $\det(J_T) > 0$ everywhere. Otherwise, it is called **orientation reversing**.

Example 8.4

The **flip diffeomorphism**

$$\begin{aligned} T : \mathbb{R}^n &\to \mathbb{R}^n \\ (x_1, \ldots, x_n) &\mapsto (-x_1, x_2, \ldots, x_n) \end{aligned}$$

is orientation reversing.

Definition 8.5. A smooth manifold M is called **orientable** if it admits a smooth atlas $\mathcal{A} = \{(U, \phi)\}$ such that all transition maps $\phi \circ \psi^{-1}$ are orientation preserving. Such an atlas gives an orientation on M. Such a pair $(M, \mathcal{A})$ is called an **oriented manifold**.

Example 8.6. $\mathbb{R}^n$ is orientable, since it admits a global chart.

Proposition 8.7

Let M be a smooth manifold of dimension n. It is orientable if and only if it admits a nowhere-vanishing differential n-form.

Note 8.8. Therefore, we can think of an oriented manifold as a pair (M, ω) where ω is a nowhere vanishing n-form on an n-dimensional manifold M.

Proof. We prove each part in turn.

- Proof of ($\Rightarrow$). Suppose we have an atlas $\{(U_\alpha, \phi_\alpha)\}$ which gives an orientation on M. We will construct a nowhere vanishing n-form on M. For any α, set $\omega_\alpha := \phi_\alpha^*(dx_1 \wedge \cdots \wedge dx_n)$. It is a nowhere vanishing n-form on U_α. Next we glue them together. For any α, β, we have

$$(\phi_\beta^{-1})^* \phi_\alpha^* (dx_1 \wedge \cdots \wedge dx_n) = \lambda_{\alpha\beta} (dx_1 \wedge \cdots \wedge dx_n).$$

Since the atlas gives an orientation on X, the functions $\lambda_{\alpha\beta}$ are always positive. Hence there holds

$$\phi_\alpha^*(dx_1 \wedge \cdots \wedge dx_n) = (\lambda_{\alpha\beta} \circ \phi_\beta) \cdot \phi_\beta^*(dx_1 \wedge \cdots \wedge dx_n).$$

This shows that at any point, ω_α is always a positive multiple of ω_β. Now we pick a partition of unity ρ_α subordinate to the open cover $\{U_\alpha\}$ and define $\omega := \sum \rho_\alpha \omega_\alpha$. At any point p, ω is a positive multiple of some ω_α. Hence ω is globally nowhere nonvanishing.

- Proof of ($\Leftarrow$). Let ω be a nowhere vanishing n-form on M. Start with an arbitrary atlas $\{(U_\alpha, \phi_\alpha)\}$ and we will modify it to be an atlas whose all transition maps are orientation preserving. For any α, consider the n-form $\phi^*(dx_1 \wedge \cdots \wedge dx_n)$. It is an n-form on U_α, hence we can write $\phi^*(dx_1 \wedge \cdots \wedge dx_n) = f_\alpha \omega$. Since both $\phi^*(dx_1 \wedge \cdots \wedge dx_n)$ and ω are nowhere zero, the function f_α is either everywhere positive or everywhere negative. If f_α is everywhere positive then we keep the chart (U_α, ϕ_α). Otherwise we modify the chart map to be $T \circ \phi_\alpha$ where T is the flip map in Example 7.2. After modifying all charts (U_α, ϕ_α), the functions f_α become always everywhere positive. A direct consequence is that all the transition maps are orientation preserving. Hence the modified atlas induces an orientation on M.

□

Lemma 8.9. Let ω, ω' be two nowhere vanishin n-form on M, then there exists a smooth function such that $\omega = f\omega'$ and f is either everywhere positive or everywhere negative.

Definition 8.10. Let ω, ω' be two nowhere vanishin n-form on M. We say ω and ω' are **equivalent** if f is positive. In this case we also say that they induce equivalent orientations on M.

Example 8.11

The two atlases $(\mathbb{R}^n, \mathrm{id})$ and $(\mathbb{R}^n, T)$ where T is the flip diffeomorphism give two different orientations on $\mathbb{R}^n$.

8.2 Integration of differential forms

Proposition 8.12 (Change of variable)

Let $f \in C^\infty(\mathbb{R}^n)$ with compact support. Let $F : \mathbb{R}^n_x \to \mathbb{R}^n_y$ be a diffeomorphism. We have the following change of variable formula

$$\int_{\mathbb{R}^n_y} f = \int_{\mathbb{R}^n_x} (f \circ F) \cdot |\det(J_F)|\,.$$

Note 8.13. We first recall this notion from multivariable calculus.

Definition 8.14. Let $\omega \in \Omega^n(\mathbb{R}^n)$ with compact support. Any such form can be written as

$$\omega = f dx_1 \wedge \cdots \wedge dx_n.$$

We define the **integral of** ω to be

$$\int \omega = \int_{\mathbb{R}^n_x} f$$

where the RHS is the Riemann integral of f.

Note 8.15. In simpler term to compute the integral of a form such as $f dx_1 \wedge \cdots \wedge dx_n$, just "erase the wedges". That is,

$$\int \omega = \int_{\mathbb{R}^n} f dx_1 \wedge \cdots \wedge dx_n = \int_{\mathbb{R}^n} dx_1 \cdots dx_n.$$

Proposition 8.16. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a diffeomorphism. We have that

$$\int F^*\omega \begin{cases} \int \omega & \text{if } F \text{ is orientation-preserving,} \\ -\int \omega & \text{if } F \text{ is orientation-reversing.} \end{cases}$$

Proof. Let y be the coordinates on the target $\mathbb{R}^n$ and x be the coordinates on the domain $\mathbb{R}^n$. Then by definition

$$\int \omega = \int_{(\mathbb{R}^n, y)} f$$

where $\omega = f \cdot dy_1 \wedge \cdots \wedge dy_n$. Then we perform change of variables formula to differential forms and Riemann integrals, respectively.

$$\int \omega = \int_{(\mathbb{R}^n, y)} f = \int_{(\mathbb{R}^n, x)} (f \circ F) \cdot |\det(J_F)|,$$

$$\int F^*\omega = \int (f \circ F) \det(J_F) dx_1 \wedge \cdots \wedge dx_n = \int_{(\mathbb{R}^n, x)} (f \circ F) \cdot \det(J_F).$$

Hence we can see the sign change directly. □

8.3 Integration on manifolds

Definition 8.17. Let M be a smooth n-manifold, and let ω be an n-form on M. Suppose first that ω with compact support, and that M admits an atlas $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}$ which gives an orientation on M. Assume the support of ω is contained in some U_α, we define the **intergal of ω over** M

$$\int_M \omega = \int_{\mathbb{R}^n} \left(\varphi_\alpha^{-1}\right)^* \omega.$$

Proposition 8.18. Let $(U_\beta, \phi_\beta) \in \mathcal{A}$ be another chart containing the support of ω, then

$$\int_{\mathbb{R}^n} (\phi_\alpha^{-1})^* \omega = \int_{\mathbb{R}^n} (\phi_\beta^{-1})^* \omega.$$

Hence the definition of $\int_X \omega$ does not depend on the choice of chart containing the support of ω.

Proof. The transition map $\phi_\alpha \circ \phi_\beta^{-1}$ preserves the orientation by our assumption. Hence the invariance follows from the proposition about the sign of integrals. □

Definition 8.19. To integrate over an entire manifold, we combine this definition with a partition of unity. Suppose M is an oriented smooth n-manifold and ω is an n-form with compact support on M. Let $\{\rho_\alpha\}$ be a partition of unity subordinate to the finite open cover $\{U_\alpha\}$. Define the **integral of** ω **over** M ot be

$$\begin{aligned}\int_M \omega &= \sum_\alpha \int_{\mathbb{R}^n} \left(\varphi_\alpha^{-1}\right)(\rho_\alpha \cdot \omega) \\ &= \sum_\alpha \int_M \rho_\alpha \omega\end{aligned}$$

Proposition 8.20. Let $\mathcal{B} = \{(U_\beta, \phi_\beta)\}$ be another atlas for M giving the same orientation as the atlas $\mathcal{A}$. Let $\{\rho_\beta\}$ be a partition of unity subordinate to $\{U_\beta\}$, then

$$\int_{M,\mathcal{A}} \omega = \int_{M,\mathcal{B}} \omega.$$

Hence we can just write them as $\int_X \omega$.

Note 8.21. This shows that integral is well defined and does not depend on the choice of the open cover or the partition of unity.

Proof. For a fixed α, we have

$$\int_X \rho_\alpha \omega = \int_X \left(\sum_\beta \rho_\beta\right) \rho_\alpha \omega = \sum_\beta \int_X \rho_\beta \rho_\alpha \omega$$

since partition of unity is a locally finite sum. Therefore we get

$$\int_{X,\mathcal{A}} \omega = \sum_\alpha \int_X \rho_\alpha \omega = \sum_\alpha \sum_\beta \int_X \rho_\beta \rho_\alpha \omega = \sum_\beta \sum_\alpha \int_X \rho_\alpha \rho_\beta \omega = \sum_\beta \int_X \rho_\beta \omega = \int_{X,\mathcal{B}} \omega.$$

The middle equality holds because that $\rho_\alpha \rho_\beta \omega$ is supported in both U_α and U_β. The integral does not depend on the choice of a single chart, by the previous proposition. $\square$

Proposition 8.22 (Properties of integral forms)
Suppose M and N are non-empty oriented smooth n-manifolds, and ω, η are n-form on M with compact support.

1. **Linearity:** If $a, b \in \mathbb{R}$, then
$$\int_M a\omega + b\eta = a\int_M \omega + b\int_M \eta.$$
2. **Orientation Reversal:** If $-M$ denotes M with the opposite orientation, then
$$\int_{-M} \omega = -\int_M \omega.$$
3. **Positivity:** If ω is a positively oriented orientation form, then $\int_M \omega > 0$.
4. **Diffeomorphism Invariance:** If $F : N \to M$ is an orientation-preserving or orientation-reversing diffeomorphism, then
$$\int_M \omega = \begin{cases} \int_N F^*\omega & \text{if } F \text{ is orientation-preserving,} \\ -\int_N F^*\omega & \text{if } F \text{ is orientation-reversing.} \end{cases}$$

8.4 Stokes' theorem

Note 8.23. The theorem will not be stated as it is not examinbale since we have not discussed manifolds with boundaries. Only its corollaries will be stated.

Theorem 8.24 (Integrals of exact forms)
Let M be an oriented smooth n-manifold. Let $\eta \in \Omega^{n-1}(M)$ with compact support. Then
$$\int_M d\eta = 0.$$

Lemma 8.25. Let $\eta \in \Omega^{n-1}(\mathbb{R}^n)$ with compact support then,
$$\int_{\mathbb{R}^n} d\eta = 0.$$

Appendix

A Topological recollections

Corollary A.1 ([Lee12, Corollary C.36.]). Suppose $U \subseteq \mathbb{R}^n$ is an open subset, and $F : U \to \mathbb{R}^n$ is a smooth function whose Jacobian determinant is nonzero at every point in U.

(a) F is an open map.

(b) If F is injective, then $F : U \to F(U)$ is a diffeomorphism.

References

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