Lie Algebra Notes

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Abstract

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Contents

1 Basic theory

Definition 1.1. An **algebra** over a field F is a vector space A over F together with a bilinear map[1]

$$\begin{aligned} A \times A &\to A \\ (x, y) &\mapsto xy. \end{aligned}$$

We say that xy is the **product** of x and y.

> **Note 1.2.** Sometimes we will refer to the bilinear map as 'multiplication'.

Example 1.3. Some very simple examples are:

1. $V = M_n(F)$ with matrix multiplication.
2. $V = \mathbb{R}^3$ with multiplication $vw = v \times w$.
3. Any vector space V with multiplication $vw = 0$ for all $v, w \in V$.

Definition 1.4. The **algebra** A is said to be **associative** if

$$(xy)z = x(yx) \quad \text{for all } x, y, z \in A.$$

Example 1.5. It is clear that $M_n(F)$ is an associative algebra, but $(\mathbb{R}^3, \times)$ is not.

1.1 Lie algebras

Definition 1.6. Let F be a field. A **Lie algebra** over F is an F-vector space L, together with a bilinear map, the **Lie bracket**

$$\begin{aligned} L \times L &\to L \\ (x, y) &\mapsto [x, y], \end{aligned}$$

satisfying the following properties:

(L1) $[x, x] = 0$ for all $x \in L$,

(L2) the Jacobi identity, $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$ for all $x, y, z \in L$.

The Lie bracket $[x, y]$ is often referred to as the **commutator** of x and y.

> **Remark 1.7.** A point on notation, sometimes to make it less cumbersome we may write $[x, y]$ as $[xy]$.

[1] A bilinear map B satisfies the following properties:

1. For any $\lambda \in F$ we have that $B(\lambda v, w) = B(v, \lambda w) = \lambda B(v, w)$.
2. The map is additive in both components. That is, $B(v_1 + v_2, w) = B(v_1, w) + B(v_2, w)$ and vice versa.

Note 1.8. Note that in the Jacobi identity we are cyclically permuting the inner Lie bracket by shifting to the right.

Proposition 1.9

If A is an associative algebra over F, then we define a bilinear operation $[-,-]$ on A by

$$[a,b] = ab - ba \quad \text{for all } a, b \in A.$$

Then A together with $[-,-]$ is a Lie algebra.

Corollary 1.10. We have that $[x,y] = -[y,x]$ for all $x, y \in L$.

Remark 1.11. This equality is not equivalent to condition L1; if the field F has characteristic 2 then this equality does not imply L1.

Proof. Consider $[x+y, x+y] = 0$. $\square$

Example 1.12. Some examples of Lie algebras.

1. $(\mathbb{R}^3, \times)$ is a Lie algebra; indeed
   (a) $v \times v = 0$ for all $v \in \mathbb{R}^3$,
   (b) and by using the BAC-CAB identity, $(a \times b) \times c = b(a \cdot c) - c(a \cdot b)$ we can prove that the multiplication satisfies the Jacobi identity.
2. Given an associative algebra with multiplication vw, we define a Lie bracket on V by $[v,w] = vw - wv$ for all $v, w \in V$. It is easy to prove this satisfies the conditions for a Lie algebra.

Example 1.13

The following are very important examples of Lie algebras.

- Suppose that V is a finite dimensional vector space over a field F. Write $\mathfrak{gl}(V)$ for the set of all linear maps from V to V. This is again a vector space over F, and it becomes a Lie algebra, known as the **general linear** (Lie) **algebra** if we define the Lie bracket by

$$[x, y] = x \circ y - y \circ x \quad \text{for all } x, y \in \mathfrak{gl}(V),$$

  where $\circ$ denotes the composition of maps.

- This is the matrix version of the above. Write $\mathfrak{gl}(n, F)$ for the vector space of all $n \times n$ matrices over F with the Lie bracket defined by

$$[x, y] = xy - yx,$$

  where xy is the usual product of matrices x and y. We note that $\dim(\mathfrak{gl}(n, F)) = n^2$.

  As a vector space, $\mathfrak{gl}(n, F)$ has a basis consisting of the matrix units e_{ij}, has a 1 in the ij-th position and all other entries are 0. We have the useful identity

$$[e_{ij}, e_{kl}] = \delta_{jk} e_{il} - \delta_{il} e_{kj}$$

  where δ is the Kronecker delta.

Note 1.14. Since a Lie algebra L is also a vector space then we by taking $V = L$ we can define the general linear Lie algebra over a Lie algebra.

Proposition 1.15

Let L be a Lie algebra. If $x, y \in L$ are such that $[x, y] \neq 0$ then x and y are linearly independent.

Proof. For the sake of contradiction, assume they are linearly dependent, so we can write $\alpha x + \beta y = 0$ where $\alpha, \beta \in F$ not all zero. By considering the Lie bracket of the sum on the LHS with y we obtain that $\alpha = 0$. It will then follow that $\beta = 0$. $\square$

1.2 Subalgebra

Definition 1.16. Given a Lie algebra L, we define a **Lie subalgebra** of L to be a vector space $K \subseteq L$ such that

$$[x, y] \in K \quad \text{for all } x, y \in K.$$

Corollary 1.17. Lie subalgebras are Lie algebras.

Proof. Trivial. $\square$

Example 1.18

We provide key examples of subalgebras of $\mathfrak{gl}(n, F)$.

- Define the **special linear** Lie algebra
$$\mathfrak{sl}(n, F) = \{A \in \mathfrak{gl}(n, F) : \mathrm{Tr}(A) = 0\}.$$
To prove this is a subalgebra we need to show that $[A, B] \in \mathfrak{sl}(n, F)$, i.e. $\mathrm{Tr}(A) = \mathrm{Tr}(B) = 0$ implies that $\mathrm{Tr}(AB - BA) = 0$. To see this note that the trace is a linear operator and that $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$. The dimension in $n^2 - 1$.

- Suppose that the characteristic of the filed is not 2. Define the **orthogonal** Lie algebra
$$\mathfrak{o}(n, F) = \left\{A \in \mathfrak{gl}(n, F) : A^\top = -A\right\}.$$
This is a Lie subalgebra as $[A, B]^\top = -[A, B]$. The dimension is $\frac{1}{2}n(n-1)$.

- Define
$$\mathfrak{t}(n, F) = \{A \in \mathfrak{gl}(n, F) : A \text{ is upper triangular}\},$$
i.e. $a_{ij} = 0$ for $i > j$. Thus, elements of $\mathfrak{t}(n, F)$ look like
$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}.$$

- Define
$$\mathfrak{u}(n, F) = \{A \in \mathfrak{t}(n, F) : A \text{ is strictly upper triangular}\},$$
i.e. $a_{ij} = 0$ for all $i \geq j$. That is the diagonal must be 0. Hence, elements of $\mathfrak{u}(n, F)$ have the following form:
$$\begin{pmatrix} 0 & a_{12} & a_{13} \\ 0 & 0 & a_{23} \\ 0 & 0 & 0 \end{pmatrix}.$$

Note 1.19. The above subalgebra inherit the Lie bracket of $\mathfrak{gl}(n, F)$.

2 Ideals and homomorphisms

2.1 Ideals

Definition 2.1. Let L be a Lie algebra. We say $I \subseteq L$ is an **ideal** of L if

1. I is a subspace of L,
2. $[I, L] \subseteq I$ where $[I, L] = \mathrm{span}\{[i, l] : i \in I, l \in L\}$.

We write $I \lhd L$ to denote that I is an ideal in L.

Note 2.2. We can interpret the second condition as

$$[i, l] \in I \quad \text{for all } i \in I \text{ and } l \in L.$$

Remark 2.3. Since $[x, y] = -[y, x]$ we do not need to distinguish between left and right ideals.

Note 2.4. Ideals are analogous to normal subgroups in group theory. Hence, we may sometimes write $I \lhd L$.

Corollary 2.5. An ideal is a subalgebra.

Proof. Apply definition. □

Example 2.6 (Converse is not true)

A subalgebra is not necessarily an ideal. For example $\mathfrak{t}(n, F)$ is a subalgebra of $\mathfrak{gl}(n, F)$, but provided $n \geq 2$, it is not an ideal. To see this, note that $e_{11} \in \mathfrak{t}(n, F)$ and $e_{21} \in \mathfrak{gl}(n, F)$. However, $[e_{21}, e_{11}] = e_{21} \notin \mathfrak{t}(n, F)$.

Example 2.7. The Lie algebra L is itself an ideal of L. At the other extreme, $\{0\}$ is an ideal of L. We call these the **trivial ideals.**

Definition 2.8. A Lie algebra is **simple** if $\dim(L) \geq 1$ and its only ideals are 0 and L.

Example 2.9

The Lie algebra $\mathfrak{sl}(2, F)$ is simple, provided the characteristic of the field is not equal to 2.

Definition 2.10. Let L be a Lie algebra and let I, J be ideals of L. We define

$$I + J = \{i + j : i \in I, j \in J\}$$

and

$$[I, J] = \operatorname{span}\{[i, j] : i \in I, j \in J\} \subseteq I \cap J.$$

Proposition 2.11

The sets $I \cap J, I + J$ and $[I, J]$ are ideals.

Proof. The proofs are mostly trivial. We note that $[I, J]$ is a subspace and for $l \in L$

$$[[i, j], l] = [[i, l], j] - [[j, l], i] \in [I, J]$$

by the Jacobi identity. □

Definition 2.12. Let L be a Lie algebra. The **centre** of L is

$$Z(L) = \{x \in L : [x, l] = 0 \text{ for all } l \in L\}$$

Definition 2.13. An **abelian Lie algebra** is a Lie algebra L in which the Lie bracket of any two elements is zero. That is, $[x, y] = 0$ for all $x, y \in L$.

Corollary 2.14. Let L be a Lie algebra. We have that $Z(L) = L$ if and only if L is abelian.

Proof. Trivial. $\square$

Example 2.15. Consider $\mathfrak{gl}(n, F)$, and let $A, B \in \mathfrak{gl}(n, F)$. We know that $[A, B] = 0$ if and only if $AB = BA$, meaning the matrices commute. However, this only occurs when A is a scalar matrix. Therefore, the centre of $\mathfrak{gl}(n, F)$ is given by

$$Z(\mathfrak{gl}(n, F)) = \{A \in \mathfrak{gl}(n, F) : A \text{ is a scalar matrix}\}.$$

Proposition 2.16

Let L be a Lie algebra then $Z(L) \triangleleft L$.

Proof. Recall that the center of a Lie algebra L is defined as

$$Z(L) = \{z \in L \mid [z, x] = 0 \text{ for all } x \in L\}.$$

We aim to show that $Z(L) \subseteq L$ is an ideal, i.e., that or all $x \in L$ and $z \in Z(L)$, the bracket $[x, z] \in Z(L)$. Let $z \in Z(L)$ and $x \in L$. By definition of the center, we have

$$[z, y] = 0 \quad \text{for all } y \in L.$$

In particular, since the Lie bracket is antisymmetric, we have:

$$[x, z] = -[z, x] = -0 = 0.$$

Thus, $[x, z] = 0$, and since $0 \in Z(L)$, it follows that $[x, z] \in Z(L)$. $\square$

2.2 Homomorphisms

Definition 2.17. Let L and M be Lie algebras over F. We say $\phi : L \to M$ is a (Lie) **homomorphism** if

1. ϕ is linear, and
2. $\phi([x, y]) = [\phi(x), \phi(y)]$ for all $x, y \in L$.

Moreover, ϕ is an **isomorphism** if it is a bijective homomorphism.

Example 2.18. If $V = F^n$ then $\mathfrak{gl}(V) \cong \mathfrak{gl}(n, F)$. To do so, we fix a basis $\mathcal{B}$ of V, then for $T \in\in \mathfrak{gl}(V)$ the map

$$\phi : T \mapsto [T]_{\mathcal{B}}$$

is a Lie homomorphism.

Proposition 2.19. Suppose $\phi : L \to M$ is a Lie algebra homomorphism then the kernel of ϕ is a subalgebra of L, and the image of ϕ is a subalgebra of M.

Proof. Similar to the one done in group theory. $\square$

Proposition 2.20. Let $\phi : L \to M$ be a Lie algebra homomorphism then the kernel of ϕ is an ideal.

Proof. We know from a previous proposition that the kernel of ϕ is a subalgebra of L. We must prove $[\ker \phi, L] \subseteq \ker \phi$. Pick $i \in \ker \phi$ and $l \in L$ then, by definition $\phi(i) = 0$. We have that $\phi([i, l]) = [\phi(i), \phi(l)] = [0, \phi(l)] \in \ker \phi$. $\square$

2.3 Adjoint

Definition 2.21. Let L be a Lie algebra, for $x \in L$, define

$$\begin{aligned} \operatorname{ad} x &: L \to L \\ (\operatorname{ad} x)(y) &\mapsto [x, y] \quad \text{for all } y \in L. \end{aligned}$$

Then $\operatorname{ad} x \in \mathfrak{gl}(L)$.

Proposition 2.22

The map

$$\begin{aligned} \operatorname{ad} : L &\to \mathfrak{gl}(L) \\ x &\mapsto \operatorname{ad} x \end{aligned}$$

is a Lie homomorphism with $\ker(\operatorname{ad}) = Z(L)$.

Proof. Since the Lie bracket is a bilinear map it follows that ad is linear. We need to check that

$$\operatorname{ad}([x, y]) = [\operatorname{ad} x, \operatorname{ad} y].$$

To do so, let $z \in L$ then,

$$\begin{aligned} [\operatorname{ad} x, \operatorname{ad} y](z) &= [x, [y, z]] - [y, [x, z]] \\ &= \operatorname{ad}([x, y])(z). \end{aligned}$$

Next, we prove the kernel is the centre. We know

$$\begin{aligned} \ker(\operatorname{ad}) &= \{x \in L : \operatorname{ad} x = 0\} \\ &= \{x \in L : (\operatorname{ad} x)(y) = 0\} \\ &= \{x \in L : [x, y] = 0 \quad \text{for all } y \in L\} \\ &= Z(L). \end{aligned}$$

□

2.4 Quotient algebras

Definition 2.23. Let L be a Lie algebra and I be an ideal of L. We can define the quotient vector space $L/I = \{x + I : x \in L\}$, where addition is defined by $(x + I) + (y + I) = (x + y) + I$, and scalar multiplication $\lambda(x + I) = (\lambda x) + I$.

Proposition 2.24. The vector space L/I is a Lie algebra with bracket

$$[x + I, y + I] = [x, y] + I$$

for all $x, y \in L$.

Proof. We check that this bracket is well-defined. Suppose □

2.4.1 The isomorphism theorem

Proposition 2.25 (First isomorphism theorem)
If $\phi : L \to M$ is a Lie homomorphism then there is a Lie isomorphism

$$L/\ker\phi \cong \mathrm{Im}\phi.$$

Proof. Consider the map $\alpha : L/\ker\phi \to \mathrm{Im}\phi$ such that $\alpha(x + \ker\phi) = \phi(x)$ for all $x \in L$. □

Example 2.26
Let L be a Lie algebra. We know $\mathrm{ad} : L \to \mathfrak{gl}(L)$ is a Lie homomorphism, and $\ker(\mathrm{ad}) = Z(L)$, so $L/Z(L) \cong \mathrm{ad}\, L$.

Example 2.27
Recall that the trace of an $n \times n$ matrix is the sum of its diagonal entries. Fix a field F and consider the linear map $\mathrm{tr} : \mathfrak{gl}(n, F) \to F$ which sends a matrix to its trace. This is a Lie algebra homomorphism, for if $x, y \in \mathfrak{gl}(n, F)$ then

$$\mathrm{tr}[x, y] = \mathrm{tr}(xy - yx) = \mathrm{tr}\, xy - \mathrm{tr}\, yx = 0,$$

so $\mathrm{tr}[x, y] = [\mathrm{tr}\, x, \mathrm{tr}\, y] = 0$. Here the first Lie bracket is taken in $\mathfrak{gl}(n, F)$ and the second in the abelian Lie algebra F.
It is not hard to see that tr is surjective. Its kernel is $\mathfrak{sl}(n, F)$, the Lie algebra of matrices with trace 0. Applying the first isomorphism theorem gives

$$\mathfrak{gl}(n, F)/\mathfrak{sl}(n, F) \cong F.$$

Proposition 2.28 (Second isomorphism theorem)
Let L be a Lie algebra and I, J ideals.

1. There exists a Lie isomorphism
$$\frac{I + J}{J} \cong \frac{I}{I \cap J}.$$
2. Ideals of the quotient L/I are of the form K/I where $I \subseteq K \subseteq L$ and K is an ideal.

Proof. We prove each statement in turn.

1. Define a map $\phi : I \to \frac{I+J}{J}$ by $\phi(i) = i + J$ for $i \in I$. This is a surjective Lie homomorphism with kernel
$$\ker\phi = \{i \in I : i + J = J\} = I \cap J.$$
Hence (1) follows from the first isomorphism theorem.

2. let M be an ideal of $\frac{L}{I}$ and define
$$K = \{x \in L : x + I \in M\}.$$
To finish the proof, check that $I \subseteq K$, K is an ideal of L, and $M = \frac{K}{I}$.

□

Proposition 2.29

Suppose that I and J are ideals of a Lie algebra L such that $I \subseteq J$. Then $J/I \triangleleft L/I$ and $(L/I)(J/I) \cong L/J$.

2.5 Derivations

Definition 2.30. Let A be an algebra over a field F. A **derivation** of A is an F-linear map $D : A \to A$ such that
$$D(ab) = aD(b) + D(a)b \quad \text{for all } a, b \in A.$$
The set of derivation is denoted by $\mathrm{Der}(A)$.

Proposition 2.31. The set $\mathrm{Der}(A)$ is a Lie subalgebra of $\mathfrak{gl}(A)$.

Proof. It is clear that $\mathrm{Der}(A)$ is a subspace. To show it is a subalgebra, we pick $D, E \in \mathrm{Der}(A)$, and we check that $[D, E] = DE - ED \in \mathrm{Der}(A)$. □

Definition 2.32. Let L be a Lie algebra. A derivation on L is a linear map $D : L \to L$ such that
$$D([x, y]) = [x, D(y)] + [D(x), y].$$

Corollary 2.33. If L is a Lie algebra, then
$$\mathrm{Im}(\mathrm{ad}) \subseteq \mathrm{Der}(L) \subseteq \mathfrak{gl}(L)$$
are both Lie subalgebra of $\mathfrak{gl}(L)$.

2.6 Structure constants

Definition 2.34. If L is a Lie algebra over a field F with basis $\{e_1, \ldots, e_n\}$, then $[-,-]$ is completely determined by the products $[e_i, e_j]$. We define the scalars $a_{ijk} \in F$ such that
$$[e_i, e_j] = \sum_{k=1}^{n} a_{ijk} e_k.$$
The a_{ijk} are the **structure constants** of L with respect to this basis.

Remark 2.35. The a_{ijk} depend on the choice of basis of L. Different bases will in general give different structure constants.

Corollary 2.36. We have that $a_{ijk} = -a_{jik}$.

Proof. Use, $[e_i, e_i] = 0$ and $[e_i, e_j] = -[e_j, e_i]$. □

Proposition 2.37

If L and M are Lie algebras over F, with bases $\mathcal{B}_1$ and $\mathcal{B}_2$, having the same structure constants, then $L \cong M$.

Proof. We prove each direction in turn.

- Proof of ($\Rightarrow$).
  Suppose $\varphi : L_1 \to L_2$ is a Lie algebra isomorphism. Let $\{e_i\}$ be a basis of L_1, and define $f_i := \varphi(e_i)$. Then $\{f_i\}$ is a basis of L_2, and for all i, j,

$$[e_i, e_j] = \sum_k c_{ij}^k e_k \quad \Rightarrow \quad [f_i, f_j] = [\varphi(e_i), \varphi(e_j)] = \varphi([e_i, e_j]) = \sum_k c_{ij}^k f_k.$$

  Hence, the structure constants in the two bases agree.

- Proof of $\Leftarrow$.
  Conversely, suppose there exist bases $\{e_i\} \subset L_1$, $\{f_i\} \subset L_2$ with

$$[e_i, e_j] = \sum_k c_{ij}^k e_k, \quad [f_i, f_j] = \sum_k c_{ij}^k f_k.$$

  Define $\varphi : L_1 \to L_2$ by $\varphi(e_i) = f_i$ and extend linearly. Then φ is a linear isomorphism and satisfies

$$\varphi([e_i, e_j]) = \sum_k c_{ij}^k f_k = [f_i, f_j] = [\varphi(e_i), \varphi(e_j)],$$

  so φ preserves the Lie bracket. Thus, φ is a Lie algebra isomorphism.

$\square$

Example 2.38
Find the structure constants of $\mathfrak{sl}(2,F)$ with respect to the basis given by the matrices

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Solution. To find the structure constants of $\mathfrak{sl}(2,F)$ with respect to $\{e,f,h\}$, we compute the commutators $[e,f]$, $[h,e]$, and $[h,f]$.

- $[e,f] = ef - fe$:

$$ef = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad fe = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}, \quad [e,f] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = h.$$

- $[h,e] = he - eh$:

$$he = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad eh = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, \quad [h,e] = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = 2e.$$

- $[h,f] = hf - fh$:

$$hf = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \quad fh = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \quad [h,f] = -2f.$$

In summary,

$$[e,f] = h, \quad [h,e] = 2e, \quad [h,f] = -2f.$$

Structure constants: $c_{12}^3 = 1$, $c_{31}^1 = 2$, $c_{32}^2 = -2$. All others are zero.

3 Low dimensional Lie algebras

Note 3.1. In this section we classify Lie algebras of dimension less than or equal to 3.

Theorem 3.2. Let L_1 and L_2 be two abelian Lie algebras. We have that L_1 and L_2 are isomorphic if and only if they have the same dimension.

Proof. We prove each direction in turn.

- Proof of $(\Rightarrow)$.
  An isomorphism of L_1 and L_2 is necessarily an isomorphism of their underlying vector spaces, so if they are isomorphic then they have the same dimension.

- Proof of $(\Leftarrow)$.
  If L_1 and L_2 have the same dimension, then there exists an invertible linear map $\phi : L_1 \to L_2$ such that
  $$\phi([x,y]) = 0 = [\phi(x), \phi(y)],$$
  since L_1 and L_2 are abelian, so their commutator is 0. Hence, ϕ is also an isomorphism of Lie algebras. $\square$

Corollary 3.3. Up to isomorphism, there exists a unique Lie algebra over any field F of dimension n for all $n \in \mathbb{N}$.

Definition 3.4. Let L be a Lie algebra. The **derived algebra** is

$$L' = \operatorname{Span}\{[x, y] : x, y \in L\}.$$

Corollary 3.5

Let L be a Lie algebra. Then L' is a Lie subalgebra of L.

Proposition 3.6. If $z \in L'$ then $\operatorname{Tr}(\operatorname{ad} z) = 0$.

Proof. As z is a linear combination of commutators $[x, y]$ with $x, y \in L$, it is sufficient to show that $\operatorname{Tr}(\operatorname{ad}[x, y]) = 0$. This is clear as

$$\operatorname{Tr}(\operatorname{ad}[x, y]) = \operatorname{Tr}[\operatorname{ad} x, \operatorname{ad} y] = \operatorname{Tr}(\operatorname{ad} x \circ \operatorname{ad} y - \operatorname{ad} y \circ \operatorname{ad} x) = 0.$$

□

Definition 3.7. If L_1 and L_2 are Lie algebras over F, their **direct sum** is

$$L_1 \oplus L_2 = \{(x_1, x_2) : x_1 \in L_1, x_2 \in L_2\}$$

equipped with the Lie bracket

$$[(x_1, x_2), (y_1, y_2)] = ([x_1, y_1], [x_2, y_2]).$$

Therefore, the set $L_1 \oplus L_2$ is a Lie algebra.

3.1 Dimension 1

Theorem 3.8 (Classification of 1-dim Lie algebras)

Any Lie algebra of dimension 1 is abelian.

Proof. Let L denote the Lie algebra, there is only one element in L, say x. Therefore, $[x, x] = 0$ for all $x \in L$. □

3.2 Dimension 2

Theorem 3.9 (Classification of 2-dim Lie algebras)

Let F be any field. Up to isomorphism there is a unique 2-dimensional non-abelian Lie algebra over F. This Lie algebra has a basis $\{x, y\}$ such that its Lie bracket is described by

$$[x, y] = x.$$

The centre of this Lie algebra is 0.

Note 3.10. We say "the Lie bracket is described by ...", this implicitly includes the information that $[x, x] = 0$ and $[x, y] = -[y, x]$.

Proof. Suppose L is a non-abelian Lie algebra of dimension 2 over a field F. If $\{u, v\}$ is a basis of L, then $[u, v] \neq 0$ as L is non-abelian. Therefore, $L' = \mathrm{Span}([u, v])$ is a 1-dimensional Lie algebra. Let $x = [u, v]$ and extend x to a basis of L i.e. $\{x, y\}$ is the new basis. Since L' is an ideal, we have that

$$[x, y] = \lambda x \quad \text{for some } \lambda \in F \setminus \{0\}.$$

Replace y by $\lambda^{-1} y$ to get $[x, y] = x$. $\square$

3.3 Dimension 3

Note 3.11. If L is a non-abelian 3-dimensional Lie algebra over a field F, then we know only that the derived algebra L' is non-zero. It might have dimension $1, 2$ or 3. Furthermore, the centre $Z(L)$ might have dimension $0, 1$ or 2. Therefore, we organise our search by relating the containment of L' to $Z(L)$.

3.3.1 The Heisenberg (Lie) algebra

Remark 3.12. This is the case where $\dim(L') = 1$ and $L' \subseteq Z(L)$.

Proposition 3.13

Up to isomorphism, there exists a unique 3-dimensional Lie algebra L over F such that $\dim(L') = 1$ and $L' \subseteq Z(L)$, namely

$$\mathfrak{u}(3, F) = \left\{ \begin{pmatrix} 0 & * & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix} \right\} \subseteq \mathfrak{gl}(3, F).$$

Proof. Take any $f, g \in L$ such that $[f, g] \neq 0$; as we have assumed that L' is 1-dimensional, the commutator $\mathrm{Span}([f, g]) = L'$. We have also assumed that $L' \subseteq Z(L)$, so we know that $[f, g]$ commutes with all elements of L. Now set

$$z := [f, g].$$

The goal is to show that $\{f, g, z\}$ is a basis of L. Since L is 3-dimensional it suffices to check that they are linearly independent. For the sake of contradiction, suppose that they are linearly dependent i.e.

$$\alpha f + \beta g + \gamma z = 0$$

for $\alpha, \beta, \gamma \in F$ not all zero. By assumption $z \in Z(L)$ so, bracketing the equation with f we obtain $\beta = 0$. Similarly, bracketing with g, we obtain $\alpha = 0$ hence, it follows that $\gamma = 0$. $\square$

Definition 3.14. We call $\mathfrak{u}(3, F)$ the **Heisenberg** (Lie) algebra over F.

Corollary 3.15. The basis of the Heisenberg algebra is given by

$$f = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, g = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Corollary 3.16

For the Heisenberg algebra we have that $\mathfrak{u}(3,F)' = Z(\mathfrak{u}(3,F))$. Hence, the center is 1-dimensional.

3.3.2 A Lie algebra L where $\dim(L') = 1$ and $L' \nsubseteq Z(L)$

Theorem 3.17

Let F be any field. Up to isomorphism, there exists a unique 3-dimensional Lie algebra over F such that $\dim(L') = 1$ and $L' \nsubseteq Z(L)$. Namely, $L_1 \oplus L_2$ where

- $\dim(L_1) = 1$, and
- $\dim(L_2) = 2$ (i.e. it is a non-abelian Lie algebra).

Proof. Pick some non-zero element $x \in L'$. Since x is not in the centre, there must be some $y \in L$ with $[x, y] \neq 0$ therefore, x and y are linearly independent. Since $\dim(L') = 1$ it follows that $L' = \operatorname{Span}\{x\}$, so

$$[x, y] = \alpha x$$

for some $\alpha \neq 0$. Replace y by $\alpha^{-1} y$ to get $[x, y] = y$. Extend $\{x, y\}$ to a basis $\{x, y, w\}$ of L. Since x spans L', there exists some scalars α, β such that

$$[x, w] = \alpha x \quad \text{and} \quad [y, w] = \beta x.$$

We claim that there exists a non-zero $z \in Z(L)$ such that $z \notin \operatorname{Span}\{x, y\}$. To show this observe, for $z = \lambda x + \mu y + \nu w \in L$, we calculate that

$$\begin{aligned} [x, z] &= [x, \lambda x + \mu y + \nu w] = \mu x + \nu \alpha x, \\ [y, z] &= [y, \lambda x + \mu y + \nu w] = -\lambda x + \nu \beta x. \end{aligned}$$

Hence, if we take $\lambda = \beta, \mu = -\alpha$ and $\nu = 1$ then $[x, y] = [y, z] = 0$ and $z \notin \operatorname{Span}\{x, y\}$. Hence, $L = \operatorname{Span}\{x, y\} \oplus \operatorname{Span}\{z\}$ is a direct sum of Lie algebras of the required form. □

SOme note in the lecture ntoes about t(2,F)

3.3.3 Lie algebras L with $\dim(L') = 2$

Note 3.18. Suppose that $\dim(L) = 3$ and $\dim(L') = 2$. We shall see that, over $\mathbb{C}$ at least, there are infinitely many non-isomorphic such Lie algebras.

Lemma 3.19

Let L be a Lie algebra. For any $x \in L'$ we have that $\mathrm{Tr}(\mathrm{ad}\ x) = 0$.

Proof. Since $L' = \mathrm{Span}\{[x, y] : x, y \in L\}$ it suffices to check the trace of $\mathrm{Tr}(\mathrm{ad}[x, y])$. Note that, $\mathrm{ad}[x, y] = [\mathrm{ad}\ x, \mathrm{ad}\ y] = \mathrm{ad}\ x\, \mathrm{ad}\ y - \mathrm{ad}\ y\, \mathrm{ad}\ x$. However, we know $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$ for any linear operator A and B, so we conclude that $\mathrm{Tr}(\mathrm{ad}[x, y]) = 0$. □

Lemma 3.20

We have the following statements.

1. The derived algebra L' is abelian.
2. For $x \in L \setminus L'$ the linear map $\mathrm{ad}\, x : L' \to L'$ is an isomorphism.

Proof. Take a basis of L', say $\{y, z\}$, and extend it to a basis of L, say by x. We prove each statement in turn.

1. It suffices to show that $[y, z] = 0$. We know that $[y, z] \in L'$, so there are scalars $\alpha, \beta \in F$ such that
$$[y, z] = \alpha y + \beta z.$$
We write the matrix of $\mathrm{ad}\, y$ with respect to the basis $\{x, y, z\}$. It has the form
$$\begin{pmatrix} 0 & 0 & 0 \\ * & 0 & \alpha \\ * & 0 & \beta \end{pmatrix}.$$
We see that $\mathrm{Tr}(\mathrm{ad}\, y) = \beta$, but we know from a previous proposition that $\mathrm{Tr}(\mathrm{ad}\, y) = 0$ hence, $\beta = 0$. Similarly, considering a matrix for $\mathrm{ad}\, z$, we get $\alpha = 0$. Thus, $[y, z] = 0$.

2. The derived algebra $L' = \mathrm{Span}([x, y], [x, z], [y, z])$ however, $[y, z] = 0$ so L' is 2-dimensional. We deduce that $\{[x, y], [x, z]\}$ is a basis of L'. Thus, the image of $\mathrm{ad}\, x$ is 2-dimensional, and $\mathrm{ad}\, x : L' \to L'$ is surjective hence, an isomorphism. □

Note 3.21. We now provide proposition that classify the Lie algebras of this form. There are two cases, which we outline in a proposition separately.

Proposition 3.22

Suppose $\dim(L) = 3, \dim(L') = 2$ and there exists $x \in L \setminus L'$ such that $\mathrm{ad}\, x : L' \to L'$ is diagonalisable. Then there exists $\mu \in F \setminus \{0\}$ and a basis $\{x, y, z\}$ of L such that

- $[x, y] = y$,
- $[x, z] = \mu z$, and
- $[y, z] = 0$.

These structure constants define a Lie algebra L_μ. We have that $L_\mu \cong L_\nu$ if and only if $\mu = \nu^{\pm 1}$.

Proof. By the first statement of the lemma above and the assumption there exists basis $\{y, z\}$ of L' such that
$$[x, y] = \lambda y \quad \text{and} \quad [x, z] = \mu z$$
and $\lambda, \mu \neq 0$. Rescale x to take $\lambda = 1$. It follows that $[y, z] = 0$ by the application of the lemma. The proof of the last assertion is omitted. □

Proposition 3.23

Suppose $\dim(L) = 3, \dim(L') = 2$ and there exists $x \in L \setminus L'$ such that $\operatorname{ad} x : L' \to L'$ is NOT diagonalisable. Then there is (up to isomorphism) a unique Lie algebra L with basis $\{x, y, z\}$ such that

- $[x, y] = y$,
- $[x, z] = y + z$, and
- $[y, z] = 0$.

Proof. Take any $x \in L \setminus L'$, as we work over $\mathbb{C}$, the map $\operatorname{ad} x : L' \to L'$ must have an eigenvector, say $0 \neq y \in L'$. We can rescale x to take $[x, y] = y$. Extend y to a basis $\{y, z\}$ of L' thus,
$$[x, z] = \lambda y + \mu z$$
where $\lambda \neq 0$ (otherwise $\operatorname{ad} x$) would be diagonalisable. By scaling z we may arrange that $\lambda = 1$. The matrix of $\operatorname{ad} x$ acting on L' therefore has the form:
$$(\operatorname{ad} x)_{y,z} = \begin{pmatrix} 1 & 1 \\ 0 & \mu \end{pmatrix}.$$
We assumed that this matrix is not diagonalisable, and therefore cannot have two distinct eigenvalues hence, $\mu = 1$. We now have that

- $[x, y] = y$,
- $[x, z] = y + z$, and
- $[y, z] = 0$.

These relations define a Lie algebra. □

3.3.4 Lie algebras L with $\dim(L) = \dim(L') = 3$

Proposition 3.24

Suppose that L is a Lie algebra over $\mathbb{C}$ of dimension 3 such that $L = L'$. Then $L \cong \mathfrak{sl}(2, \mathbb{C})$.

Remark 3.25. Therefore, there is only one such Lie algebra.

Note 3.26. Recall, the dimension of $\mathfrak{sl}(n, F)$ is $n^2 - 1$.

Remark 3.27. This proposition is not true over any field F.

Proof. The proof is split into multiple steps.

Step 1: Let $x \in L$ be non-zero. We claim that $\operatorname{ad} x : L \to L$ has rank 2 (i.e. the image is 2-dimensional). Extend x to a basis of L, say $\{x, y, z\}$. Then $L' = \operatorname{Span}\{[x, y], [x, z], [y, z]\}$. But $L' = L$, so this set must be linearly independent, and hence the image of $\operatorname{ad} x$ has a basis $\{[x, y], [x, z]\}$ of size 2 as required.

Step 2: We claim that there is some $h \in L$ such that $\operatorname{ad} h : L \to L$ has an eigenvector with a non-zero eigenvalue. Choose any non-zero $x \in L$. If $\operatorname{ad} x$ has a non-zero eigenvalue, then we may take $h = x$. If $\operatorname{ad} x : L \to L$ has non-zero eigenvalue, then, as it has rank 2 (by the above claim), its Jordan canonical form must be

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

This matrix indicates there is a basis of L extending $\{x\}$, say $\{x, y, z\}$ such that $[x, y] = x$ and $[x, z] = y$. So $\operatorname{ad} y$ has x as an eigenvector with eigenvalue -1, and we may take $h = y$.

Step 3: By the previous step, we may find $h, x \in L$ such that $[h, x] = \alpha x \neq 0$. Since $h \in L$ and $L = L'$, we know from Exercise 2.5 that $\operatorname{ad} h$ has trace zero. It follows that $\operatorname{ad} h$ must have three distinct eigenvalues $\alpha, 0, -\alpha$. If y is an eigenvector for $\operatorname{ad} h$ with eigenvalue $-\alpha$, then $\{h, x, y\}$ is a basis of L. In this basis, $\operatorname{ad} h$ is represented by a diagonal matrix.

Step 4: To fully describe the structure of L, we need to determine $[x, y]$. Note that

$$[h, [x, y]] = [[h, x], y] + [x, [h, y]] = \alpha[x, y] + (-\alpha)[x, y] = 0.$$

We now make two applications of Step 1. Firstly, $\ker \operatorname{ad} h = \operatorname{Span}\{h\}$, so $[x, y] = \lambda h$ for some $\lambda \in \mathbb{C}$. Secondly, $\lambda \neq 0$, as otherwise $\ker \operatorname{ad} x$ is 2-dimensional. By replacing x with $\lambda^{-1} x$, we may assume that $\lambda = 1$. So now

- $[h, x] = 2x$,
- $[h, y] = -2y$, and
- $[x, y] = h$.

These are the structure constants for the basis e, f, h of $\mathfrak{sl}(2, \mathbb{C})$. $\square$

4 Solvable Lie algebras

Remark 4.1. We make a key distinction in terminology here. In the U.K. it is common to use the word "solvable" in place of "solvable". But we will use the American counterpart to remain in line with key texts used.

Note 4.2. Abelian algebras are easily understood. There is a sense in which some Lie algebras we outlined in the previous section are close to being abelian. In this section, we ask to what extent can we "approximate" a Lie algebra by abelian Lie algebras.

Lemma 4.3

Suppose that I is an ideal of L. Then L/I is abelian if and only if I contains the derived algebra L'.

Note 4.4. This lemma tells us that the derived algebra L' is the smallest ideal of L with an abelian quotient. By the same argument, the derived algebra L' itself has a smallest ideal whose quotient is abelian, namely the derived algebra of L', which we denote $L^{(2)}$, and so on. This motivates the next definition.

Proof. The algebra L/I is abelian if and only if for all $x, y \in L$ we have

$$[x+I, y+I] = [x, y] + I = I$$

or, equivalently, for all $x, y \in L$ we have $[x, y] \in I$. Since I is a subspace of L, this holds if and only if the space spanned by the brackets $[x, y]$ is contained in I; that is, $L' \subseteq I$. $\square$

Definition 4.5. Define the following

$$\begin{aligned} L^{(1)} &= L' = [L, L] \\ L^{(2)} &= \left[L^{(1)}, L^{(1)}\right] \\ &\vdots \\ L^{(k)} &= \left[L^{(k-1)}, L^{(k-1)}\right]. \end{aligned}$$

Then the series

$$L \supseteq L^{(1)} \supseteq L^{(2)} \supseteq \cdots$$

such that $L^{(i)}/L^{(i+1)}$ is abelian (by the above lemma), is the **derived series** of L.

Corollary 4.6. $L^{(k)}$ is an ideal of L (and not just an ideal of $L^{(k-1)}$).

Proof. The product of ideals is an ideal. $\square$

Definition 4.7. A Lie algebra is **solvable** if $L^{(m)} = 0$ for some m.

Example 4.8

Some examples.

1. The Heisenberg algebra is solvable.
2. Similarly, the algebra of upper triangular matrices, $\mathfrak{t}(n, F)$ and $\mathfrak{u}(n, F)$ are solvable.
3. Any 2-dimensional Lie algebra is solvable.
4. $\mathfrak{sl}(2, \mathbb{C})$ is NOT solvable, because $\mathfrak{sl}(2, \mathbb{C})' = \mathfrak{sl}(2, \mathbb{C})$, by the arguments below it cannot be solvable.

Example 4.9. Let L be a Lie algebra.

1. If $\dim L = 2$, then we have $\dim L' \leq 1$ (from the classification from a previous section), so $L^{(2)} = 0$ and L is solvable.
2. If $\dim L = 3$ and $\dim L' \leq 2$, then by (1) we have $L^{(3)} = 0$, and hence L is solvable.
3. If $\dim L = 3$ and $\dim L' = 3$, then $L^{(1)} = L$, so $L^{(i)} = L$ for all i, and L is not solvable.

Proposition 4.10

Let L be a Lie algebra.

1. If L is solvable, then all subalgebras of L and quotient algebras of L/I are solvable.
2. If I is an ideal, and both I and L/I are solvable then L is solvable.
3. If I, J are solvable ideals of L then $I + J$ are solvable.

Proof. We prove each statement in turn.

1. Suppose $L^{(m)} = 0$. For a subalgebra M we have that $M^{(m)} = 0$, so M is also solvable. For an ideal I we have

   $$(L/I)^{(i)} = (L^{(i)} + I)/I$$

   and hence $(L/I)^{(m)} = 0$ so L/I is solvable.
2. Assume $I^{(m)} = 0$ and $(L/I)^{(n)} = 0$ then

   $$(L/I)^{(n)} = (L^{(n)} + I)/I = I$$

   so $L^{(n)} \subseteq I$. Hence

   $$0 = \left(L^{(n)}\right)^{(m)} = L^{(n+m)}$$

   so L is solvable.

3. By the second isomorphism theorem $(I + J)/I \cong J/I \cap J$, so it is solvable by applying the first statement of the proposition. Since I is also solvable the second statement implies that $I + J$ is solvable.

$\square$

Corollary 4.11. Let L be a finite dimensional Lie algebra. Then L has a unique solvable ideal which contains every solvable ideal of L

Proof. Let R be a solvable ideal of the largest possible dimension. Suppose that I is any solvable ideal. By Proposition 5.10(3), we know that $R + I$ is a solvable ideal. Now $R \subseteq R + I$ and hence $\dim R \leq \dim(R + I)$. We chose R of maximal possible dimension, and therefore we must have $\dim R = \dim(R + I)$ and hence $R = R + I$, so I is contained in R. $\square$

Definition 4.12. The unique maximal solvable ideal of finite dimensional Lie algebra L is called the **radical** of L, written $\mathrm{Rad}(L)$.

Example 4.13

Some examples.

1. If L is solvable, then $\mathrm{Rad}(L) = L$.
2. Take $L = \mathfrak{sl}(2, \mathbb{C})$. Here, $L = L'$ (by considering the structure constants $[ef] = h$, $[eh] = -2e$, $[fh] = 2f$, we note that any basis vector is contained in L'), so L is not solvable. Also, L is simple, so $\mathrm{Rad}(L)$ is 0 or L. Therefore, $\mathrm{Rad}(L) = 0$.
3. Take $L = \mathfrak{gl}(2, \mathbb{C})$. Here $\mathrm{Rad}(L) = Z(L) = \{\lambda I : \lambda \in \mathbb{C}\}$.

Definition 4.14. Let L be a Lie algebra. If $\mathrm{Rad}(L) = 0$ we call L a **semisimple** Lie algebra.

Proposition 4.15. Let L be a Lie algebra of dimensional greater than 1. If L is simple then L is semisimple.

> **Remark 4.16.** We need a constraint on the dimension otherwise the abelian Lie algebra of dimension 1 would be a counterexample to this statement.

Proof. If L is simple, then $L' = 0$ or L. If $L' = 0$ this means L is abelian so, the commutator $[x, y] = 0 \in V$ for any subspace V. Thus, any subspace becomes an ideal which is a contradiction as we assume L to be simple. Hence, $L' = L$, which means L is not solvable. Also, $\mathrm{Rad}(L) = 0$ or L, and is not L, because L is not solvable. Therefore, $\mathrm{Rad}(L) = 0$, and L is semisimple. $\square$

Proposition 4.17

If L is a Lie algebra, the factor algebra $L/\mathrm{Rad}(L)$ is semisimple.

Proof. Let $\overline{J}$ be a solvable ideal of $L/\operatorname{rad} L$. By the ideal correspondence, there is an ideal J of L containing $\operatorname{rad} L$ such that $\overline{J} = J/\operatorname{rad} L$. By definition, $\operatorname{rad} L$ is solvable, and $J/\operatorname{rad} L = \overline{J}$ is solvable by hypothesis. Therefore, the proposition above implies that J is solvable. But then J is contained in $\operatorname{rad} L$; that is, $\overline{J} = 0$. Therefore, $\operatorname{Rad}(L/\operatorname{Rad}(L)) = 0$. □

5 Nilpotent Lie algebras

Definition 5.1. Define a series of ideals of a Lie algebra L:

$$\begin{aligned} L^1 &= L' = [L, L], \\ L^2 &= [L, L^1] \\ &\vdots \\ L^k &= [L, L^{k-1}]. \end{aligned}$$

We have $L \supseteq L^1 \supseteq L^2 \supseteq \cdots$, the **lower central series** of L.

> **Remark 5.2.** The reason for the name "central series" comes from the fact that $L^k/L^{k+1} \subseteq Z\left(L/L^{k+1}\right)$.

Corollary 5.3. L^k is an ideal of L (and not just an ideal of L^{k-1}).

Proof. The product of ideals is an ideal. □

Definition 5.4. The Lie algebra L is said to be **nilpotent** if for some $m \geq 1$ we have $L^m = 0$.

Corollary 5.5. If L is a nilpotent Lie algebra, then L is solvable.

Proof. Suppose L is nilpotent. Then the *lower central series* terminates at zero after finitely many steps, i.e., there exists $n \in \mathbb{N}$ such that $L^{(n)} = 0$. We claim that $L^{(k)} \subseteq L^k$ for all $k \geq 1$. That is, the derived series is contained in the lower central series. We prove this by induction.
Base case: For $k = 1$, we have $L^{(1)} = [L, L] = L^1$, so $L^{(1)} \subseteq L^{(1)}$.
Inductive step: Assume $L^{(k)} \subseteq L^k$. Then

$$L^{(k+1)} = [L^{(k)}, L^{(k)}] \subseteq [L^k, L^k] \subseteq [L, L^k] = L^{k+1}$$

Hence, by induction, $L^{(k)} \subseteq L^k$ for all k. Since $L^n = 0$, it follows that $L^{(n)} \subseteq L^n = 0$, so $L^{(n)} = 0$. Therefore, the derived series terminates at zero and L is solvable. □

Example 5.6

Some examples.

1. Suppose $\dim L = 2$, L is non-abelian. By a proposition, L has basis x, y with $[xy] = x$. So $L^1 = \operatorname{Span}(x), L^2 = \operatorname{Span}(x), \ldots, L^i = \operatorname{Span}(x)$ for all i. Hence, L is **not** nilpotent.

2. The Lie algebra $\mathfrak{u}(n, F)$ is nilpotent, but $\mathfrak{t}(n, F)$ is not nilpotent (for $n \geq 2$).

Lemma 5.7

Let L be a Lie algebra.

(a) If L is nilpotent, then any Lie subalgebra and quotient Lie algebra of L is nilpotent.

(b) If $L/Z(L)$ is nilpotent, then L is nilpotent.

Proof. Part (a) is clear from the definition. By induction, one can show that $(L/Z(L))^k$ is equal to $(L^k + Z(L))/Z(L)$. So if $(L/Z(L))^m$ is zero, then L^m is contained in $Z(L)$ and therefore $L^{m+1} = 0$. □

5.1 Ascending central series

Lemma 5.8. Let L be a non-zero nilpotent Lie algebra. Then $Z(L) \neq \emptyset$.

Proof. Let n be the largest natural number such that $L^n \neq 0$. Then $[L, L^n] = L^{n+1} = 0$, so $L^n \subseteq Z(L)$. But L^n is non-zero, so the same holds for $Z(L)$, too. □

Example 5.9

The claim is not true when L is merely solvable. Let $L = \langle x, y\rangle$ be the 2-dimensional solvable Lie algebra subject to the relation $[xy] = y$. If $ax + by \in Z(L)$ for some $a, b \in \mathbb{C}$, then

$$[ax + by, x] = -by, \quad [ax + by, y] = ay,$$

so both a and b must vanish. Therefore $Z(L) = 0$.

Definition 1.3. For every Lie algebra L we are going to define the ascending central series of L, a chain of ideals:

$$0 = L_0 \subseteq L_1 \subseteq \cdots \subseteq L_n \subseteq \cdots$$

as follows. Set $L_1 = Z(L)$. If $L_{n-1} \lhd L$ is already defined, then let $q_{n-1} : L \to L/L_{n-1}$ be the quotient map. We set $L_n = q_{n-1}^{-1}(Z(L/L_{n-1}))$. Since it is the preimage of an ideal, the subset $L_n \subseteq L$ is an ideal.

Lemma 5.10

A Lie algebra L is nilpotent if and only if $L_n = L$ for some n.

Proof. First assume that L is nilpotent. It will be sufficient to show that for every $n \geq 1$ the ideal L_{n-1} is strictly contained by L_n unless $L_{n-1} = L$; since L is finite dimensional, this shows that the strictly increasing sequence $0 = L_0 \subseteq L_1 \subseteq \cdots$ must terminate. Since L/L_{n-1} is the quotient of a nilpotent Lie algebra, it is nilpotent. Since $L_{n-1} \neq L$, this quotient is non-zero, so by Lemma 1.1 its centre is non-zero. So the preimage of the latter is strictly larger than L_{n-1}.

We are going to prove the converse by induction on $\dim(L)$. The case $\dim(L) = 0$ is trivially true. Now assume that $L \neq 0$. Note that $Z(L) \neq 0$; if $L_1 = Z(L) = 0$, we would get by induction that $L_n = 0$, a contradiction. Therefore $\dim(L/L_1) < \dim(L)$. Since $L_n = q_1^{-1}(L/L_1)_{n-1}$ for every $n \geq 1$, we get that $(L/L_1)_n = L/L_1$ for some n, and hence L/L_1 is nilpotent by the induction hypothesis. Therefore $(L/L_1)^m = 0$ for some m, which implies that $L^m \subseteq L_1 = Z(L)$. But

$$L^{m+1} = [L, L^m] \subseteq [L, Z(L)] = 0,$$

so L is nilpotent.

□

6 Engel's theorem

Definition 6.1. A linear map $T : V \to V$ is **nilpotent** if $T^r = 0$ for some r.

Corollary 6.2. Some corollaries:

- A nilpotent linear transformation $T : V \to V$ has only eigenvalue value 0, so its characteristic polynomial x^n, where $n = \dim(V)$. Hence, $T^n = 0$ by the Cayley-Hamilton Theorem.
- The Jordan canonical form of T is of the form

$$\begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix}$$

  i.e. strictly upper triangular. Such a matrix is also nilpotent (as its characteristic polynomial is x^n), and lies in $\mathfrak{u}(n, F)$.
- Any Lie subalgebra of $\mathfrak{u}(n, F)$ consists of nilpotent matrices.

Theorem 6.3 (Engel's theorem)

Let V be an n-dimensional vector space over F. Suppose L is a Lie subalgebra of $\mathfrak{gl}(V)$ such that every element of L is nilpotent (i.e. $T^n = 0$ for all $T \in L$ and some $n \in \mathbb{N}$). Then there exists a basis $\mathcal{B}$ of V such that every $T \in L$ is represented by a strictly upper triangular matrix. That is,

$$L \cong \{[T]_{\mathcal{B}} \text{ is strictly upper triangular} : T \in L\} \subseteq \mathfrak{u}(n, F).$$

In particular, L is a nilpotent Lie algebra.

Note 6.4. This theorem is converse to the third statement of the above corollary.

The idea of the proof

To prove Engel's Theorem, we adapt the strategy used to prove the analogous result for a single nilpotent linear transformation. The proof of this result is outlined in the following.

Let V be an n-dimensional vector space where $n \geq 1$, and let $T : V \to V$ be a nilpotent linear map.

(i) There is a non-zero vector $v \in V$ such that $T(v) = 0$.

(ii) Let $U = \mathrm{Span}(v)$. We have that T induces a nilpotent linear transformation $\overline{T} : V/U \to V/U$. By induction, we know that there is a basis $\{v_1 + U, \dots, v_{n-1} + U\}$ of V/U in which $\overline{T}$ has a strictly upper triangular matrix. Therefore, $\{v, v_1, \dots, v_{n-1}\}$ is a basis of V and that the matrix of x in this basis is strictly upper triangular.

The crucial step in the proof is the analogue of part (i): we must fine a non-zero vector $v \in V$ that lies in the kernel of <u>every</u> $T \in L$.

We will build the necessary results in lemmas and propositions.

Lemma 6.5. Let $L \subset \mathfrak{gl}(V)$ be a Lie subalgebra. Let $x \in L$, and suppose $x : V \to V$ is nilpotent. Then $\operatorname{ad} x : L \to L$ is also nilpotent.

Proof. Let $y \in L$ then

$$(\operatorname{ad} x)(y) = [x, y] = xy - yx$$

hence,

$$\begin{aligned}(\operatorname{ad} x)^2(y) &= (\operatorname{ad} x)(xy - yx) \\ &= [x, xy - yx] \\ &= x(xy - yx) - (xy - yx)x \\ &= x^2y - 2xyx + yx^2.\end{aligned}$$

Similarly, $(\operatorname{ad} x)^3$ is a linear combination of terms of the form x^3y and so on. In general, $(\operatorname{ad} x)^m(y)$ is a linear combination of terms of the form $x^j y x^{m-j}$ where $0 \leq j \leq m$ (this can be proved by induction on m).

By hypothesis, $x^n = 0$ for some n. Then for $m = 2n$, either x^j or x^{m-j} is 0 for all $0 \leq j \leq m$. This shows that $(\operatorname{ad} x)^{2n} = 0$, hence, $\operatorname{ad} x$ is nilpotent. □

Lemma 6.6. Let $L \subset \mathfrak{gl}(V)$ be a subalgebra, and I an ideal of L. Define

$$W = \{v \in V : T(v) = 0 \text{ for all } T \in I\}.$$

Then W is L-invariant. That is, $S(W) \subseteq W$ for all $S \in L$.

Proof. Let $S \in L$ and $w \in W$/ For each $T \in I$ notice that $[T, S] = TS - ST$, so

$$TS(w) = ST(w) + [T, S](w) = 0 + 0$$

since both $T, [T, S] \in I$. Therefore, $S(w) \in W$. Since the choice of w was arbitrary we have $S(W) \subseteq W$. □

Proposition 6.7

Let $L \subset \mathfrak{gl}(V)$ be a subalgebra consisting of nilpotent elements. Then there exists a vector $0 \neq v \in V$ such that $T(v) = 0$ for all $T \in L$.

Proof. We proceed by induction on $\dim(L)$. If $\dim(L) = 1$, then $L = \mathrm{Span}(T)$ where $T : V \to V$ is a nilpotent linear transformation, which we know from a previous observation has an eigenvector v such that $T(v) = 0$.

Step 1. Take a maximal Lie subalgebra $A \subset L$. We claim that A is an ideal of L and that $\dim(A) = \dim(L) - 1$. Consider the quotient L/A. We define a linear map

$$\begin{aligned}\phi : A &\to \mathfrak{gl}(L/A) \\ \phi(a)(x + A) &= [a, x] + A \\ &= (\text{ad } a)(x) + A.\end{aligned}$$

This is well-defined, for if $x \in A$ then $[x, a] \in A$. Moreover, ϕ is a Lie homomorphism, for if $a, b \in A$ then

$$\begin{aligned}[\phi(a), \phi(b)](x + A) &= \phi(a)([b, x] + A) - \phi(b)([a, x] + A) \\ &= ([a, [b, x]] + A) - ([b, [a, x]] + A) \\ &= [a, [b, x]] - [b, [a, x]] + A \\ &= [[a, b], x] + A\end{aligned}$$

by the Jacobi identity. The last term is equal to $\phi([a, b])(x + A)$, as required.

So $\phi(A) \subset \mathfrak{gl}(L/A)$ is a subalgebra and $\dim \phi(A) < \dim L$. To apply the inductive hypothesis, we need to know that $\phi(a)$ is a nilpotent linear transformation of L/A. But $\phi(a)$ is induced by $\operatorname{ad} a$; by a lemma from above, we know that $\operatorname{ad} a : L \to L$ is nilpotent and therefore $\phi(a)$ is as well.

By the inductive hypothesis, there is some non-zero element $y + A \in L/A$ such that $\phi(a)(y + A) = [a, y] + A = 0$ for all $a \in A$. That is, $[a, y] \in A$ for all $a \in A$. Set $\tilde{A} := A \oplus \text{Span}\{y\}$. This is a Lie subalgebra of L containing A. By maximality, $\tilde{A}$ must be equal to L. Therefore $L = A \oplus \text{Span}\{y\}$. As A is an ideal in $\tilde{A}$, it follows that A is an ideal of L.

Step 2. We now apply the inductive hypothesis to $A \subseteq \text{gl}(V)$. This gives us a non-zero $w \in V$ such that $a(w) = 0$ for all $a \in A$. Hence

$$W = \{v \in V : a(v) = 0 \text{ for all } a \in A\}$$

is a non-zero subspace of V.

By a lemma above, W is invariant under L, so in particular $y(W) \subseteq W$. Since y is nilpotent, the restriction of y to W is also nilpotent. Hence there is some non-zero vector $v \in W$ such that $y(v) = 0$. We may write any $x \in L$ in the form $x = a + \beta y$ for some $a \in A$ and some $\beta \in F$. Doing this, we have

$$x(v) = a(v) + \beta y(v) = 0.$$

This shows that v is a non-zero vector in the kernel of every element of L. □

6.1 Proof of Engel's theorem

Proof. Let $L \subset \mathfrak{gl}(V)$ be a subalgebra consisting of nilpotent elements. we aim to show that there exists a basis $\mathcal{B}$ such that $\{[T]_{\mathcal{B}} : T \in L\} \subset \mathfrak{u}(n, F)$. Let $n = \dim(V)$. We proceed by induction on n. If $V = \{0\}$ then there is nothing to do so, we assume $\dim(V) \geq 1$. By the above proposition there exists $0 \neq v \in V$ such that $T(v) = 0$ for all $T \in L$. Let $U = \text{Span}(v)$, any $T \in L$ induced a linear transformation

$$\begin{aligned}\overline{T} : V/U &\to V/U \\ \overline{T}(x + U) &= T(x) + U\end{aligned}$$

for all $x \in V$. Moreover, since T is nilpotent so is $\overline{T}$. The map

$$\begin{aligned} L &\to \mathfrak{gl}(V/U) \\ T &\mapsto \overline{T} \end{aligned}$$

is a Lie homomorphism, and its image is a Lie subalgebra of $\mathfrak{gl}(V/U)$ consisting of nilpotent. As $\dim(V/U) = \dim(V) - \dim(U) = n - 1$, we can apply the inductive hypothesis to obtain a basis

$$\overline{\mathcal{B}} = \{v_1 + U, \ldots, v_{n-1} + U\}$$

of V/U such that

$$\left\{ \left[\overline{T}\right]_{\overline{\mathcal{B}}} : T \in L \right\} \subseteq \mathfrak{u}(n-1, F).$$

Then

$$\mathcal{B} = \{v, v_1, \ldots, v_{n-1}\}$$

is a basis of V, and

$$\{[T]_{\mathcal{B}} : T \in L\} \subseteq \mathfrak{u}(n, F).$$

□

6.2 Engel Ver. 2

Theorem 6.8 (Engel's theorem version 2)
A (finite-dimensional) Lie algebra L is nilpotent if and only if $\operatorname{ad} x : L \to L$ is nilpotent for all $x \in L$.

Note 6.9. The first version of Engel's theorem is about subalgebras of $\mathfrak{gl}(V)$, this version is about abstract Lie algebras.

Remark 6.10. It is very tempting to assume that a Lie subalgebra L of $\mathfrak{gl}(V)$ is nilpotent if and only if there is a basis of V such that the elements of L are represented by strictly upper triangular matrices. However, the "⇒" direction is false. For example, any 1-dimensional Lie algebra is (trivially) nilpotent. Let I denote the identity map in $\mathfrak{gl}(V)$. The Lie subalgebra $\operatorname{Span}(I)$ of $\mathfrak{gl}(V)$ is therefore nilpotent. In any basis of V, the map I is represented by the identity matrix, which is certainly not strictly upper triangular.

Proof. We prove each direction in turn.

- Proof of (⇒).
  Recall that L being nilpotent means $L^m = 0$ for some m, more explicitly:

  $$[x_0, [\ldots, [x_{m-2}[x_{m-1}, x_m]] \ldots]] = 0$$

  for all $x_i \in L$. Hence, $(\operatorname{ad} x)^m = 0$.

- Proof of ($\Leftarrow$).
  Recall that $\operatorname{ad}: L \to \mathfrak{gl}(L)$ is a Lie homomorphism, so its image is a Lie subalgebra of $\mathfrak{gl}(L)$ consisting of nilpotent elements $\operatorname{ad} x$. BY the first version of Engel's theorem, $\operatorname{Im}(\operatorname{ad}) = \operatorname{ad}(L)$ is nilpotent and also $\ker(\operatorname{ad}) = Z(L)$. By the first isomorphism theorem we have
  $$\operatorname{ad}(L) = L/Z(L).$$
  Suppose $(\operatorname{ad}(L))^m = 0$, then $0 = (\operatorname{ad}(L))^m = \frac{L^m + Z(L)}{Z(L)}$ hence $L^m \leq Z(L)$. So, $L^{m+1} = [L, L^m] \subseteq [L, Z(L)] = 0$ which implies L is nilpotent.

□

7 Lie's theorem

Note 7.1. Let L be a Lie subalgebra of $\mathfrak{gl}(V)$. We would now like to understand when there is a basis of V such that the elements of L are all represented by upper triangular matrices.

Theorem 7.2 (Lie's theorem)
Let V be an n-dimensional vector space over V and let L be a solvable Lie subalgebra of $\mathfrak{gl}(V)$. Then there exists a basis $\mathcal{B}$ of V such that
$$L \cong \{[T]_B \text{ is triangular} : T \in L\} \subseteq \mathfrak{t}(n, \mathbb{C}).$$

Remark 7.3. The theorem is false for field of prime characteristic.

Idea of proof

When $\dim L = 1$, $L = \operatorname{Span}(T)$, then the theorem says that there exists a basis B such that $[T]_B$ is upper-triangular. This is easily proved by induction on $\dim V$:

(a) There exists an eigenvector $0 \neq v \in V$ for T (since the field is $\mathbb{C}$).

(b) Let $U = \operatorname{Span}(v)$. Then T induces $\overline{T} : V/U \to V/U$ by $\overline{T}(v + U) = T(v) + U$ and by the inductive hypothesis, there exists a basis $\overline{\mathcal{B}} = (v_1 + U, \dots, v_{n-1} + U)$ of V/U such that $[\overline{T}]_{\overline{\mathcal{B}}}$ is upper-triangular. Then $\mathcal{B} = \{v, v_1, \dots, v_{n-1}\}$ is a basis of V, and $[T]_{\mathcal{B}}$ is upper-triangular.

As for the proof of Engel's Theorem, the main part of the proof of Lie's Theorem is Step (a): finding a common eigenvector for all T in the Lie algebra L.

7.1 Weights

Note 7.4. In linear algebra, one is often interested in the eigenvalues and eigenvectors of a fixed linear map. We now generalise these notions to families of linear maps. Let A be a subalgebra of $\mathrm{gl}(V)$. It seems reasonable to say that $v \in V$.

Proposition 7.5. Let V be a vector space over F. Let $A \subseteq \mathfrak{gl}(V)$ be a subalgebra, and suppose there exists $0 \neq v \in V$ such that v is a common eigenvector for all $a \in A$. So,

$$a(v) = \lambda(a)v$$

for all $a \in A$, where $\lambda(a) \in F$. Then the function $\lambda : A \to F$ is linear.

Definition 7.6. A **weight** for a Lie subalgebra $A \subseteq \mathfrak{gl}(V)$ is a linear map $\lambda : A \to F$ such that

$$V_\lambda = \{v \in V : a(v) = \lambda(a)v \text{ for all } a \in A\} \neq \{0\}\,.$$

Such a non-zero subspace V_λ is called the **weight space** for the weight λ.

Example 7.7

Let $L = d(n, F)$ be the set of diagonal matrices. Let $e_1, \ldots, e_n$ be the standard basis of F^n. Then for

$$x = \begin{pmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_n \end{pmatrix} \in L,$$

we have $x(e_i) = \alpha_i e_i$. So $\mathrm{Span}(e_i)$ is a weight space for L, with weight $\lambda_i : L \to F$, where

$$\lambda_i \begin{pmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_n \end{pmatrix} = \alpha_i.$$

Lemma 7.8 (Invariance lemma)

Assume thtat F has characteristic zero. Let $L \subseteq \mathfrak{gl}(V)$ be a subalgebra and let A be an ideal of L. Let $\lambda : A \to F$ be a weight of A. The associated weight space V_λ is an L-invariant subspace of V (i.e. $l(V_\lambda) \subseteq V_\lambda$ for all $l \in L$).

Proof. Omitted because I find it too long. $\square$

Proposition 7.9

Let V be a finite-dimensional vector space over $\mathbb{C}$. Let $L \subseteq \mathfrak{gl}(V)$ be a solvable Lie subalgebra. Then there exists $0 \neq v \in V$ such that v is a common eigenvector of all $x \in L$ (i.e. $x(v) \in \mathrm{Span}(v)$ for all $x \in L$).

Proof. We proceed by induction on $\dim L$. The statement is true for $\dim L = 1$, i.e. $L = \mathrm{Span}(T)$, as T has an eigenvector (since the field is $\mathbb{C}$).

Now, assume that $\dim L > 1$. As L is solvable, $L' \subset L$. Choose a subspace A of L such that

$$L' \subseteq A \text{ and } \dim A = \dim L - 1.$$

Let $z \in L \setminus A$, so

$$L = A \oplus \mathrm{Span}(z) \quad \text{(as vector spaces)}.$$

Then $[AA] \subseteq L' \subseteq A$, and $[A, z] \subseteq L' \subseteq A$, so A is an ideal of L. By the inductive hypothesis, there exists $0 \neq w \in V$ such that w is a common eigenvector for all $a \in A$. So

$$a(w) = \lambda(a)w \text{ for all } a \in A.$$

Let

$$V_\lambda = \{v \in V : a(v) = \lambda(a)v \text{ for all } a \in A\}.$$

Then $w \in V_\lambda$, so $V_\lambda \neq 0$, and hence λ is a weight of A. By Proposition 5.2, V_λ is L-invariant, so

$$z(V_\lambda) \subseteq V_\lambda.$$

Therefore, there exists $0 \neq v \in V_\lambda$ such that v is an eigenvector for z; say $z(v) = \beta v$. Then for any $a + \alpha z \in L$ where $a \in A$, $\alpha \in \mathbb{C}$, we have

$$(a + \alpha z)(v) = \lambda(a)v + \alpha\beta v.$$

Hence v is a common eigenvector for all $x \in L$. □ □

8 Representation theory

8.1 Basics

Definition 8.1. Let L be a Lie algebra over a field F. A **representation** of L is a Lie algebra homomorphism

$$\rho : L \to \mathfrak{gl}(V)$$

where V is a finite-dimensional vector space over F.

Note 8.2. For brevity, we will sometimes omit mention of the homomorphism and just say that V is a representation of L.

Definition 8.3. A representation ρ is **faithful** if $\ker(\rho) = \{0\}$.

Corollary 8.4. If ρ is a faithful representation of L we have that $L \cong \mathrm{Im}(\rho)$, a Lie subalgebra of $\mathfrak{gl}(V)$.

Definition 8.5. If we fix a basis $\mathcal{B}$ of V, the map

$$\begin{aligned} L &\to \mathfrak{gl}(n, F) \\ x &\mapsto [\rho(x)]_{\mathcal{B}} \end{aligned}$$

for all $x \in L$, is called a **matrix representation** of L.

Example 8.6

Some examples of representations.

1. For any Lie algebra L over F, the zero map $L \to F$ sending $l \mapsto 0$ for all $l \in L$ is the **trivial** (1-dimensional) representation of L.
2. If L is a Lie subalgebra of $\mathfrak{gl}(V)$ for some vector space V, the inclusion map $\iota : L \to \mathfrak{gl}(V)$ is a representation of L, called the **natural representation** of L.
3. If L is any Lie algebra, the map

$$\begin{aligned} \operatorname{ad} : L &\to \mathfrak{gl}(L) \\ x &\to \operatorname{ad} x \end{aligned}$$

is a representation, called the **adjoint representation** of L. Its kernel is $Z(L)$.

Example 8.7. More examples.

1. Let $L = \mathfrak{sl}(2, F)$ with basis

$$\mathcal{B} = \left\{ e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \right\}$$

and structure constants $[e, f] = h$, $[e, h] = -2e$, $[f, h] = 2f$. With respect to this basis, the adjoint representation of L gives a representation of L, sending

$$e \mapsto (\operatorname{ad}\, e)_{\mathcal{B}} = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad h \mapsto (\operatorname{ad}\, h)_{\mathcal{B}} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \quad f \mapsto (\operatorname{ad}\, f)_{\mathcal{B}} = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}.$$

Now, $Z(L) = 0$, provided $\operatorname{char}(F) \neq 2$, so L is isomorphic to the Lie subalgebra of $\mathfrak{gl}(3, F)$, spanned by these 3 matrices.

2. Let L be the 2-dimensional non-abelian Lie algebra over F, with basis $\{x, y\}$ and structure constant $[x, y] = x$. Define $\rho : L \to \mathfrak{gl}(2, F)$ by

$$\rho(x) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \rho(y) = \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix},$$

and extend it linearly to L, i.e. for all $\alpha, \beta \in F$

$$\rho(\alpha x + \beta y) = \begin{pmatrix} -\beta & \alpha + \beta \\ 0 & 0 \end{pmatrix}.$$

One can easily check that ρ is a representation of L. (It suffices to check that $[\rho(x), \rho(y)] = \rho(x)$, which is clear when we multiply out the matrices.)

Note 8.8. To "extend linearly" means to define a map on all elements of a vector space by specifying its action on a basis and then applying linearity to extend it to any linear combination of basis elements.

8.2 Modules for Lie algebras

Definition 8.9. If $\rho : L \to \mathfrak{gl}(V)$ is a representation, it is notationally convenient to drop the ρ an instead of writing $\rho(l)v$ we write lv. This defines a map

$$\begin{aligned} L \times V &\to V \\ (l, v) &\mapsto \rho(l)(v) = lv \end{aligned}$$

satisfying

1. $(l_1 + l_2)v = l_1 v + l_2 v$,
2. $l(v_1 + v_2) = lv_1 + lv_2$,
3. $\lambda(lv) = l(\lambda v) = (\lambda l)v$,
4. $[l_1, l_2]v = l_1(l_2 v) - l_2(l_1 v)$

for all $l_i, l \in L, v_i, v \in V$ and $\lambda \in F$.

Definition 8.10. Let L be a Lie algebra over a field F. A vector space V over F is an **L-module** if there exists a mpa

$$\begin{aligned} L \times V &\to V \\ (l, v) &\mapsto lv \end{aligned}$$

satisfying all the conditions above.

Example 8.11

Any representation of L gives an L-module. In fact, L-modules and representations of L are equivalent concepts:

- Given a representation $\rho : L \to \mathfrak{gl}(V)$, defining $lv = \rho(l)(v)$ for $l \in L$ and $v \in V$ makes V an L-module.
- Given an L-module V, we can define $\rho : L \to \mathfrak{gl}(V)$ by

  $$\rho(l)(v) = lv$$

  for all $l \in L$ and $v \in V$. Then the conditions of an L-module imply that ρ is a Lie homomorphism, hence a representation of L.

Example 8.12

Some examples of Lie modules.

1. Let L be a 2-dimensional Lie algebra, with basis x, y and $[xy] = x$. We have a matrix representation $\rho : L \to \mathfrak{gl}(2, F)$ sending

$$x \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad y \mapsto \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}.$$

The corresponding L-module is $V = F^2$ with the standard basis e_1, e_2 and multiplication

$$xe_1 = 0, \quad xe_2 = e_1,$$

$$ye_1 = -e_1, \quad ye_2 = e_1.$$

2. Let L be any Lie algebra. We have the **adjoint representation** $\operatorname{ad} : L \to \mathfrak{gl}(L)$. The corresponding L-module, the **adjoint module**, is L itself, with multiplication

$$lv = (\operatorname{ad} l)(v) = [l, v] \quad \text{for all } l \in L, v \in L.$$

8.3 Submodules

Definition 8.13. Let L be a Lie algebra and V and L-module. A subspace $W \subseteq V$ is a **submodule** if W is L-invariant. That is, $lW \subseteq W$ for all $l \in L$.

Example 8.14

Ideals of L are submodules.

Proposition 8.15

Let L be a solvable finite-dimensional Lie algebra over $\mathbb{C}$, and let V be a finite dimensional non-zero L-module. Then V has a 1-dimensional submodule.

Proof. Let $\rho : L \to \mathfrak{gl}(V)$ be the corresponding representation of L that is, $\rho(l)(v) = lv$ for all $l \in L$ and $v \in V$. Then,

$$\frac{L}{\ker \rho} \cong \operatorname{im} \rho \subseteq \mathfrak{gl}(V).$$

Now, $\operatorname{im}(\rho)$ is a solvable Lie subalgebra of $\mathfrak{gl}(V)$ by some previous proposition hence, by another previous proposition there exists a common eigenvector $0 \neq v \in V$ for all $\rho(l) \in \operatorname{im} \rho$. This implies

$$lv = \rho(l)(v) \in \operatorname{Span}(v)$$

for all $l \in L$. Hence $\operatorname{Span}(v)$ is a 1-dimensional submodule. □

Example 8.16. In the previous example $\operatorname{Span}(e_1)$ was such a submodule.

8.4 Irreducible modules

Definition 8.17. An L-module V is said to be **irreducible** if

- it is non-zero, and
- the only submodules are $\{0\}$ and V.

We then also say that the corresponding representation $\rho : L \to \mathfrak{gl}(V)$ is **irreducible**.

Example 8.18

Some examples.

1. If V is 1-dimensional, then V is irreducible. For example, the trivial representation is always irreducible.
2. If L is the adjoint module, the submodules are ideals of L. So the adjoint L-module is irreducible if and only if L is a simple Lie algebra (or $\dim L = 1$). For example, the adjoint representation $\mathfrak{sl}(2, \mathbb{C}) \to \mathfrak{gl}(3, \mathbb{C})$ is irreducible.

Corollary 8.19

Let L be a solvable Lie algebra over $\mathbb{C}$. Then every irreducible L-module is 1-dimensional.

Proof. Follows immediately from the previous proposition. □

8.5 Quotient modules

Definition 8.20. Let L be a Lie algebra, V an L-module and W a submodule. We can turn the quotient vector space V/W into an L-module, called the **quotient module** by defining

$$l(v + W) = lv + W$$

for all $l \in L$ and $v \in V$.

Proposition 8.21. The quotient module is an L-module.

Proof. As usual, we must first check that the multiplication is well-defined. Suppose that $v + W = v' + W'$. Then $(lv) + W - (lv') + W = l(v - v') + W = 0 + W$ as $v - v' \in W$ and W is L-invariant. The remaining properties are easy to check. □

8.6 Composition Series

Definition 8.22. Let L be a Lie algebra and V a (finite-dimensional) nonzero L-module. Choose a non-zero submodule $V_1 \subset V$ of minimal dimension. Then V_1 is irreducible. Form the quotient $\frac{V}{V_1} \neq 0$. If $\frac{V}{V_1} \neq 0$, find a submodule $\frac{V_2}{V_1}$ of minimal dimension, where $V_2 \subset V$ is a submodule $V_1 \subset V_2$. Then $\frac{V_2}{V_1}$ is irreducible. Continuing, the sequence

$$0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_r = V,$$

where $\frac{V_i}{V_{i-1}}$ are irreducible, terminates, because V is finite-dimensional. This is a **composition series** of V.

8.7 Direct sums

Definition 8.23. Let V be a vector space over a field F, and let $U, W \subseteq V$ be subspaces. We say that V is the **direct sum** of U and W, written

$$V = U \oplus W,$$

if the following two conditions hold:

- Every vector $v \in V$ can be written as
$$v = u + w \quad \text{for some } u \in U,\ w \in W.$$
- This expression is unique, i.e.,
$$U \cap W = \{0\}.$$

Definition 8.24. If V is an L-module such that $V = U \oplus W$, where both U and V are L-submodules of V, we say that V is the **direct sum** of the L-module U and W.

> **Remark 8.25.** More generally, let V be an L-module. Suppose $V = U_1 \oplus \cdots \oplus U_r$, i.e., $V = U_1 + \cdots + U_r$ and $U_i \cap \left(\sum_{j \neq i} U_j\right) = 0$ for all i, where each U_i is a submodule. Then we say V is the *direct sum* of the submodules $U_1, \ldots, U_r$.

Definition 8.26. The L-module V is said to be **completely reducible** if it can be written as a direct sum of irreducible L-modules; that is $V = S_1 \oplus S_2 \oplus \cdots \oplus S_k$, where each S_i is an irreducible L-module.

Example 8.27

Some examples.

1. Let $L = d(2, F) = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} : \alpha, \beta \in F \right\}$, an abelian Lie algebra. Let $V = F^2$, the natural L-module with standard basis e_1, e_2. Then $V_1 = \mathrm{Span}(e_1), V_2 = \mathrm{Span}(e_2)$ are submodules, as
$$\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} e_1 = \alpha e_1, \quad \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} e_2 = \beta e_2,$$
and $V = V_1 \oplus V_2$. Each V_i is irreducible, so V is completely reducible.
2. Let $L = t(2, F)$, $V = F^2$ be the natural L-module. The only nontrivial (not 0 and not V) submodule of V is $\mathrm{Span}(e_1)$, so V is not completely reducible.

8.8 Homomorphisms

Definition 8.28. Let L be a Lie algebra, and let V and W be L-modules. An **L-module homomorphism** is a linear map

$$\phi : V \to W \quad \text{such that} \quad \phi(lv) = l\phi(v)$$

for all $l \in L$ and $v \in V$.

Definition 8.29. An **isomorphism** is a bijective L-module homomorphism.

We want to interpret the notion of homomorphism in terms of representations. Let $\rho_V : L \to \mathfrak{gl}(V)$ and $\rho_W : L \to \mathfrak{gl}(W)$ be the corresponding representations of L. Then ϕ is a homomorphism if

$$\phi(\rho_V(l)(v)) = \rho_W(l)(\phi(v)) \text{ for all } l \in L, v \in V$$

i.e. $\phi \circ \rho_V(l) = \rho_W(l) \circ \phi$ for all $l \in L$. If ϕ is an isomorphism, this says

$$\rho_W(l) = \phi \circ \rho_V(L) \circ \phi^{-1}.$$

So we can choose bases B_V of V and B_W of W such that the corresponding matrices are equal:

$$[\rho_V(l)]_{B_V} = [\rho_W(l)]_{B_W} \text{ for all } l \in L,$$

i.e. the matrix representations

$$l \mapsto [\rho_V(l)]_{B_V}, \quad l \mapsto [\rho_W(l)]_{B_W}$$

are **identical**.

Note 8.30. This is a way to tell whether two L-modules are isomorphic.

Example 8.31

Let $L = \text{Span}(x)$, a 1-dimensional Lie algebra. Then L has matrix representations

$$\rho_1 : x \mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \rho_2 : x \mapsto \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}.$$

The corresponding L-modules are isomorphic as the matrices $\rho_1(x), \rho_2(x)$ are similar.

9 Representations of $\mathfrak{sl}(2, \mathbb{C})$

Note 9.1. We shall use the following basis of $\mathfrak{sl}(2, \mathbb{C})$:

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

and structure constants

$$[e, g] = h, \ [h, e] = 2e \ [h, f] = -2f$$

9.1 The modules V_d

Note 9.2. We begin by constructing a family of irreducible representations of $\mathfrak{sl}(2, \mathbb{C})$.

Definition 9.3. Let $\mathbb{C}[X, Y]$ be the vector space of all polynomials in X, Y with complex coefficients. For each integer $d \geq 0$, let V_d be the subspace of homogenous polynomials in X and Y of degree d i.e.

$$V_d = \text{Span}\left(X^d, X^{d-1}Y, \dots, XY^{d-1}, Y^d\right).$$

Thus $\dim(V_d) = d + 1$.

Note 9.4. Recall homogenous polynomials of degree d are polynomials whose non-zero terms have all the same degree.

Example 9.5

V_0 is the 1-dimensional vector space of constant polynomilas, and for $d \geq 1$, the space V_d has a basis of monomials $\left\{X^d, X^{d-1}Y, \ldots, XY^{d-1}, Y^d\right\}$.

Definition 9.6. We now make V_d into an $\mathfrak{sl}(2, \mathbb{C})$-module. Define the folllwoign Lie algebra homomorphism

$$\phi : \mathfrak{sl}(2, \mathbb{C}) \to \mathfrak{gl}(V_d)$$

by

$$\begin{aligned}
\phi(e) &= X\frac{\partial}{\partial Y} \\
\phi(f) &= Y\frac{\partial}{\partial X} \\
\phi(h) &= X\frac{\partial}{\partial X} - Y\frac{\partial}{\partial Y}
\end{aligned}$$

Theorem 9.7

With these definitions, ϕ is a representation of $L = \mathfrak{sl}(2, \mathbb{C})$. Hence V_d is an L-module of dimensional $d + 1$.

Proof. By construction ϕ is lienar. Thus, all we have to check is that ϕ preserveds the Lie brackets. By linearity, it is enough to check this on the basis elements of $\mathfrak{sl}(2, \mathbb{C})$, so there are three equations to verify:

1. $[\phi(e), \phi(f)] = \phi([e, f]) = \phi(h)$.
2. $[\phi(h), \phi(e)] = 2\phi(e)$.
3. $[\phi(h), \phi(f)] = -\phi(f)$.

This is a computational exercises, we only show the first one. We begin by showing $[\phi(e), \phi(f)] = \phi([e, f]) = \phi(h)$. If we apply the left-hand side to a basis vector X^aY^b with $a, b \geq 1$ and $a + b = d$, we get

$$\begin{aligned}
[\phi(e), \phi(f)](X^aY^b) &= \phi(e)\left(\phi(f)(X^aY^b)\right) - \phi(f)\left(\phi(e)(X^aY^b)\right) \\
&= \phi(e)\left(aX^{a-1}Y^{b+1}\right) - \phi(f)\left(bX^{a+1}Y^{b-1}\right) \\
&= a(b+1)X^aY^b - b(a+1)X^aY^b \\
&= (a-b)X^aY^b.
\end{aligned}$$

This is the same as $\phi(h)(X^aY^b)$. We check separately the action on X^d,

$$\begin{aligned}
[\phi(e), \phi(f)](X^d) &= \phi(e)\left(\phi(f)(X^d)\right) - \phi(f)\left(\phi(e)(X^d)\right) \\
&= \phi(e)\left(dX^{d-1}Y\right) - \phi(f)(0) = dX^d,
\end{aligned}$$

which is the same as $\phi(h)(X^d)$. Similarly, one checks the action on Y^d, so $[\phi(e), \phi(f)]$ and $\phi(h)$ agree on a basis of V_d and so are the same linear map.

$\square$

9.1.1 Matrix representation

It can be useful to know the matrices that correspond to the action of e, f, h on V_d; these give the matrix representation corresponding to ϕ. As usual, we take the basis $\mathcal{B} = \left\{X^d, X^{d-1}Y, \ldots, Y^d\right\}$ of V_d. The calculations in the proof of the theorem above show that the matrix of $\phi(e)$ with respect to this basis is

$$e \mapsto [\phi(e)]_{\mathcal{B}} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & d \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

the matrix of $\phi(f)$ is

$$f \mapsto [\phi(f)]_{\mathcal{B}} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ d & 0 & \cdots & 0 & 0 \\ 0 & d-1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix},$$

and $\phi(h)$ is diagonal:

$$h \mapsto [\phi(h)]_{\mathcal{B}} = \begin{pmatrix} d & 0 & \cdots & 0 & 0 \\ 0 & d-2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & -d+2 & 0 \\ 0 & 0 & \cdots & 0 & -d \end{pmatrix}.$$

where the diagonal entreis are the numbers $d-2k$, where $k = 0, 1, \ldots, d$.

Example 9.8

Let $L = \mathfrak{sl}(2, \mathbb{C})$. The module V_0 is the trivial representation $e, f, g \mapsto 0$. The module V_1 is given by

$$e \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and it is equal to the natural L-module. The module V_2 is given by:

$$e \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}, \quad h \mapsto \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \quad f \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

and is isomorphic to the adjoint L-module.

9.1.2 Irreducibility

Theorem 9.9

The $\mathfrak{sl}(2, \mathbb{C})$-module V_d is irreducible.

Proof. Suppose U is a non-zero submodule of V_d. Then $\phi(e), \phi(f), \phi(h)$ send U to U. Since $\phi(h)$ has $d+1$ distinct eigenvalues, the eigenvalues of the restriction $\phi(h)|_U : U \to U$ are also distinct, and U contains an eigenvector for $\phi(h)$. The eigenspaces of $\phi(h)$ are the 1-dimensional spaces spanned by the basis vectors in B. Hence

$$X^a Y^b \in U \text{ for some } a, b.$$

Apply $\phi(e)$ to this successively to get

$$X^{a+1}Y^{b-1},\ X^{a+2}Y^{b-2},\ \ldots,\ X^d \in U,$$

and apply $\phi(f)$ to this successively to get

$$X^{a-1}Y^{b+1},\ X^{a-2}Y^{b+2},\ \ldots,\ Y^d \in U.$$

Hence $U = V_d$. This shows V_d is irreducible. □

9.2 Classifying the irreducible $\mathfrak{sl}(2, \mathbb{C})$-modules

Note 9.10. It is clear that for different d the $\mathfrak{sl}(2, \mathbb{C})$-modules V_d cannot be isomorphic, as they have different dimensions. In this section, we prove that any finite-dimensional irreducible $\mathfrak{sl}(2, \mathbb{C})$-module is isomorphic to one of the V_d.

Remark 9.11. For brevity, we shall write $e^2 v$ rather than $e(ev)$, and so on.

Lemma 9.12. Let V be a finite dimension $\mathfrak{sl}(2, \mathbb{C})$-module.

1. If $v \in V$ with $hv = \lambda v$ then

$$h(ev) = (\lambda + 2)ev, \quad h(fv) = (\lambda - 2)fv.$$

2. The L-module V contains an eigenvector $w \neq 0$ for h such that $ew = 0$.

Proof. We prove each statement in turn.

1. Observe that

$$h(ev) = e(hv) + [he]v = e(\lambda v) + 2ev = (\lambda + 2)ev,$$

and similarly for $h(fv)$.

2. Note that the linear map $v \mapsto hv$ has an eigenvector (since the field is $\mathbb{C}$). Say $hv = \lambda v$. Consider

$$v, ev, e^2 v, \ldots.$$

If all of them are non-zero, by (1), these are eigenvectors for h with distinct eigenvalues, hence they are linearly independent. Hence, as V is finite-dimensional, there exists k such that $e^k v \neq 0$, but $e^{k+1} v = 0$. Put $w = e^k v$ to complete the proof.

□

Theorem 9.13

If V is a finite dimensional irreducible $\mathfrak{sl}(2, \mathbb{C})$ module, then $V \cong V_d$ for some d.

Proof. Let V be an irreducible finite-dimensional $\mathfrak{sl}(2, \mathbb{C})$-module. By Lemma the second part of the lemma above, there exists $w \neq 0$ such that

$$hw = \lambda w, \quad ew = 0.$$

By the proof of of the lemma above, there exists d such that

$$f^d w \neq 0, \quad f^{d+1} w = 0.$$

Step 1. The elements $w, fw, \ldots, f^d w$ form a basis of V, consisting of h-eigenvectors with eigenvalues $\lambda, \lambda - 2, \ldots, \lambda - 2d$.

Indeed, by the lemma above, these are eigenvectors for h with the given eigenvalues, hence they are linearly independent. To show they span V, we set

$$U = \operatorname{Span}(w, fw, \ldots, f^d w).$$

We show U is a submodule. Well, clearly

$$fU \subseteq U, \quad hU \subseteq U.$$

We will show that $ef^k w \in \operatorname{Span}(w, fw, \ldots, f^{k-1} w)$ for all $k \leq d$ by induction on k. This is clearly true for $k = 0$, as $ew = 0$. Assume this is true for $k - 1$. Then

$$ef^k w = e(f(f^{k-1} w)) = (fe + [e, f]) f^{k-1} w = (fe + h) f^{k-1} w.$$

By the inductive hypothesis, $ef^{k-1} w \in \operatorname{Span}(w, \ldots, f^{k-2} w)$, and hence

$$fef^{k-1} w \in \operatorname{Span}(w, \ldots, f^{k-1} w).$$

This completes the induction and shows that $eU \subseteq U$. We have hence shown that U is a submodule of V and, as V is irreducible, we have $U = V$.

Step 2. If B is the basis in (Step 1), then

$$[h]_B = \begin{pmatrix} \lambda & & & \\ & \lambda - 2 & & \\ & & \ddots & \\ & & & \lambda - 2d \end{pmatrix}.$$

Also, $h = [e, f] \in L'$, so $\operatorname{Tr}[h]_B = 0$. Hence

$$\lambda + (\lambda - 2) + \cdots + (\lambda - 2d) = 0.$$

Thus $(d + 1)\lambda = d(d + 1)$, which shows that $\lambda = d$.

Step 3. We have that $V \cong V_d$.

The L-module V has a basis $w, fw, \dots, f^d w$, and the L-module V_d has a basis $X^d, fX^d, \dots, f^d X^d$. Both bases consist of h-eigenvectors with eigenvalues $d, d-2, \dots, -d$. Define $\varphi : V \to V_d$ by

$$\varphi(f^k w) = f^k X^d \quad \text{for } 0 \leq k \leq d.$$

We show that φ is an isomorphism of L-modules. We need to show that

$$\varphi(lv) = l\varphi(v) \quad \text{for all } v \in V$$

for $l = e, f, h$. For f:

$$f\varphi(f^k w) = f(f^k X^d) = f^{k+1} X^d = \varphi(f^{k+1} w).$$

For h:

$$h\varphi(f^k w) = h(f^k X^d) = (d - 2k) f^k X^d = \varphi(h(f^k w)).$$

For e, we show that

$$e\varphi(f^k w) = \varphi(e f^k w)$$

by induction on k. For $k = 0$, we have

$$e\varphi(w) = eX^d = 0 = \varphi(ew).$$

This completes the induction and shows that $V \cong V_d$. □

10 Cartan's criteria

Note 10.1. In this section, we describe a practical way to decide whether a Lie algebra is semisimple or, solvable by looking at traces of linear maps.

Remark 10.2. A trick:

$$\mathrm{Tr}([a, b]c) = \mathrm{Tr}(a[b, c])$$

for linear transformations a, b, c of a vector space.

10.1 Jordan normal form

Note 10.3. In this section we recall about the Jordan normal (or canonical) form.

Let V be a finite-dimensional vector space over $\mathbb{C}$ and let $f : V \to V$ be a linear map, then there si a basis $\mathcal{B}$ of V in which f is given by the direct sum of Jordan blocks i.e.

$$[f]_{\mathcal{B}} = \begin{pmatrix} J_{t_1}(\lambda_1) & 0 & \cdots & 0 \\ 0 & J_{t_2}(\lambda_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_{t_k}(\lambda_k) \end{pmatrix},$$

where

$$J_t(\lambda) = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1 & \cdots & 0 & 0 \\ 0 & 0 & \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{pmatrix}_{t\times t}.$$

10.2 Jordan decomposition

Note 10.4. Working over the complex numbers allows us to consider the Jordan normal form of a linear transformation.

We can rewrite the matrix $[f]_{\mathcal{B}}$ as $D+N$ where

$$D = \operatorname{diag}(\lambda_1, \ldots, \lambda_1, \lambda_2, \ldots, \lambda_k, \ldots, \lambda_k)$$

and

$$N = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix}.$$

which is strictly upper triangular. Clearly N is a nilpotent matrix (i.e. $N^l = 0$ for some l), and we have $DN = ND$.

Theorem 10.5 (JNF theorem)

Any linear transformation f of a vector space V over $\mathbb{C}$ has a **Jordan decomposition**, $f = d + n$ where $d : V \to V$ is a diagonalisable linear transformation and $n : V \to V$ isa nilpotent linear transformation such that $dn = nd$.

Definition 10.6. A **Jordan basis** $\mathcal{B}$ of V is a basis in which $[d]_{\mathcal{B}} = D$ and $[n]_{\mathcal{B}} = N$.

Lemma 10.7. Let f have Jordan decomposition $f = d + n$ as above, where d is diagonalisable, n is nilpotent, and d, n commute.

1. There is a polynomial $p(X) \in \mathbb{C}[X]$ such that $p(x) = d$.
2. Fix a basis of V in which d is diagonal. Let $\bar{d}$ be the linear map whose matrix with respect to this basis is the complex conjugate of the matrix of d. There is a polynomial $q(X) \in \mathbb{C}[X]$ such that $q(x) = \bar{d}$.

Proof. We prove each statement in turn.

- Observe that $(X-\lambda_i)^{a_i}$, for $i=1,\dots,k$, are pairwise coprime. The Chinese Remainder Theorem says that the natural map
$$\mathbb{C}[X] \to \bigoplus_{i=1}^{k} \mathbb{C}[X]/((X-\lambda_i)^{a_i})$$
is surjective.
Hence, for $\lambda_1,\dots,\lambda_k$, we can find a polynomial $p(X)$ such that
$$p(X) \equiv \lambda_i \mod (X-\lambda_i)^{a_i}, \text{ for } i=1,2,\dots,k.$$
Then we have that
$$p(X) = \lambda_i + \varphi_i(X)(X-\lambda_i)^{a_i}, \text{ for some } \varphi_i(X) \in \mathbb{C}[X],$$
and hence for $v \in V_i$, we have $p(x)v = \lambda_i v + \varphi(x)(x-\lambda_i I)^{a_i} v = \lambda_i v$. Therefore,
$$p(x) = \bigoplus_{i=1}^{k} \lambda_i I_{V_i} = d.$$

- Define $q(X)$ to be the polynomial equivalent to $\overline{\lambda}_i$ modulo $(X-\lambda_i)^{a_i}$, for $i=1,\dots,k$. The same argument now shows that $q(x)=\overline{d}$.

□

Lemma 10.8. Let V be a vector space, and suppose that $x \in \mathfrak{gl}(V)$ has Jordan decomposition $d+n$. We have that $\mathrm{ad}(x) : \mathfrak{gl}(V) \to \mathfrak{gl}(V)$ has Jordan decomposition $\mathrm{ad}(d)+\mathrm{ad}(n)$.

Proof. By linearity of ad, we have $\mathrm{ad}(x)=\mathrm{ad}(d)+\mathrm{ad}(n)$. Next, a previous lemma, $\mathrm{ad}(n)$ is nilpotent, because n is nilpotent. We claim that $\mathrm{ad}(d)$ is diagonalisable. Consider the basis in which d is diagonal. Recall that E_{ij} (the matrix with 1 in the ij entry and 0 everywhere else) is the standard basis of $\mathfrak{gl}(V)$. For this basis of $\mathfrak{gl}(V)$, $\mathrm{ad}(d)$ is diagonal. Finally, $[\mathrm{ad}(d), \mathrm{ad}(n)] = \mathrm{ad}([d,n]) = 0$. By the Jordan canonical form theorem, $\mathrm{ad}(d)$ and $\mathrm{ad}(n)$ are the diagonalisable and the nilpotent parts of $\mathrm{ad}(x)$, respectively.

□

10.3 Testing for solvability

Proposition 10.9

Let V be a finite-dimensional vector space over $\mathbb{C}$, and let $L \subseteq \mathfrak{gl}(V)$ be a solvable Lie algebra. Then $\mathrm{Tr}(xy)=0$ for all $x \in L$ and $y \in L'$.

Proof. By Lie's Theorem, there is a basis B of V such that $[x]_B$ is an upper-triangular matrix for all $x \in L$. Then $[x_1]_B[x_2]_B - [x_2]_B[x_1]_B$ is strictly upper-triangular for all $x_1, x_2 \in L$. Hence for any $y \in L'$, $[y]_B$ is strictly upper-triangular, hence $[x]_B[y]_B$ is strictly upper-triangular. Therefore, $\mathrm{Tr}(xy) = \mathrm{Tr}([x]_B[y]_B) = 0$. □ □

Lemma 10.10. Let V be a finite-dimensional vector space over $\mathbb{C}$. If $x,y,z : V \to V$ are linear maps, then $\mathrm{Tr}([x,y]z) = \mathrm{Tr}(x[y,z])$.

Proof. We have that

$$\mathrm{Tr}([x,y]z) = \mathrm{Tr}(xyz - yxz) = \mathrm{Tr}(xyz) - \mathrm{Tr}(yxz)$$

and

$$\mathrm{Tr}(x[y,z]) = \mathrm{Tr}(xyz - xzy) = \mathrm{Tr}(xyz) - \mathrm{Tr}(xzy),$$

so we only have to check that $\mathrm{Tr}(yxz) = \mathrm{Tr}(xzy)$. But this is clear, since $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$:

$$\mathrm{Tr}(yxz) = \mathrm{Tr}(y(xz)) = \mathrm{Tr}((xz)y) = \mathrm{Tr}(xzy).$$

□

Proposition 10.11

Let V be a finite dimensional vector space over $\mathbb{C}$. Let $L \subseteq \mathfrak{gl}(V)$ be a Lie subalgebra. If $\mathrm{Tr}(xy) = 0$ for $x, y \in L$, then L is solvable.

Proof. The idea is to show that every $x \in L'$ is nilpotent. Then Engel's Theorem implies that L' is a nilpotent algebra. Hence L' is solvable, therefore L is solvable.

Let $x \in L'$. Consider the Jordan decomposition $x = d + n$. There exists a basis B of V with respect to which $d = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$ and $[n]_B$ is strictly upper-triangular. We need to prove that $d = 0$, i.e. $\lambda_i = 0$ for all i. This will follow if we show that

$$\sum_{i=1}^{n} \lambda_i \lambda_i = 0.$$

Define $\overline{d} : V \to V$ by $[\overline{d}]_B = \mathrm{diag}(\overline{\lambda}_1, \dots, \overline{\lambda}_n)$. Let us note that

$$\mathrm{Tr}(\overline{d}x) = \sum_{i=1}^{n} \lambda_i \overline{\lambda}_i.$$

It is enough to prove that

$$\mathrm{Tr}(\overline{d}[yz]) = 0$$

for any $y, z \in L$, because $x \in L' = \mathrm{Span}\{[yz] : y, z \in L\}$. By a previous lemma

$$\mathrm{Tr}(\overline{d}[yz]) = \mathrm{Tr}([\overline{d}y]z).$$

By a previous lemma, the Jordan decomposition of $\mathrm{ad}(x)$ is $\mathrm{ad}(d) + \mathrm{ad}(n)$. By a previous lemma, there is a polynomial $q(X) \in \mathbb{C}[X]$ such that $\mathrm{ad}(\overline{d}) = q(\mathrm{ad}(x))$. But it is clear that $\mathrm{ad}(\overline{d}) = \mathrm{ad}(\overline{d})$. Therefore, $\mathrm{ad}(\overline{d})$ maps $L \to L$ (because this is a polynomial in $\mathrm{ad}(x)$ and $\mathrm{ad}(x) : L \to L$). In particular, $\mathrm{ad}(\overline{d})(y) = [\overline{d}, y] \in L$. Since $\mathrm{Tr}(xy) = 0$ for any $x, y \in L$, we conclude that $\mathrm{Tr}([\overline{d}y]z) = 0$. We have seen above that this implies $\lambda_1 = \cdots = \lambda_n = 0$, so that x is nilpotent. □

Proposition 10.12

A Lie algebra L is solvable if and only if $\mathrm{ad}(L)$ is solvable.

Remark 10.13. We note that $\mathrm{ad}(L) = \{\mathrm{ad}\, x : x \in L\} \subseteq \mathfrak{gl}(L)$.

Proof. We have that $\mathrm{ad}(L) = L/Z(L)$. Since $Z(L)$ is always abelian hence solvabe it must be that L is solvable if and only if $\mathrm{ad}(L)$ is solvable. □

10.4 The Killing form

Definition 10.14. Let L be a Lie algebra over $\mathbb{C}$. The **Killing form** is the map

$$\begin{aligned} K : L \times L &\to \mathbb{C} \\ (x, y) &\mapsto \mathrm{Tr}(\mathrm{ad}(x) \circ \mathrm{ad}(y)) \end{aligned}$$

Proposition 10.15. The Killing form is a symmetric bilinear map.

Proof. The Killing form is bilinear because ad is linear, the composition of maps is bilinear, and tr is linear. It is symmetric because $\mathrm{tr}(ab) = \mathrm{tr}(ba)$ for linear maps a and b. □

Corollary 10.16. Another very important property of the Killing form is its *associativity*, which states that for all $x, y, z \in L$ we have

$$K([x, y], z) = K(x, [y, z]).$$

Proof. Follows immediately from a previous lemma. □

Theorem 10.17 (Cartan's first criterion)
Let L be a Lie algebra over $\mathbb{C}$. We have that L is solvable if and only if $K(L, L') = 0$.

Proof. We prove each direction in turn.

- Proof of ($\Rightarrow$).
  Consider $\mathrm{ad} : L \to \mathfrak{gl}(L)$. Now, $\mathrm{ad}(L)$ is a Lie subalgebra of $\mathfrak{gl}(V)$. By the remark above, the solvability of L is equivalent to the solvability of $\mathrm{ad}(L)$. We are given that L is solvable, hence $\mathrm{ad}(L) \subseteq \mathfrak{gl}(V)$ is solvable. Thus $\mathrm{Tr}(\mathrm{ad}(x)\,\mathrm{ad}(y)) = 0$ for any $x \in L$ and any $y \in L'$ by a previous proposition.

- Proof of ($\Leftarrow$).
  We are given that $\mathrm{Tr}(\mathrm{ad}(x) \circ \mathrm{ad}(y)) = 0$ for any $x \in L, y \in L'$. By a previous proposition we obtain that $L^{(2)}$ is solvable. But this implies that L' is solvable so L is solvable.

□

10.4.1 Recollections of linear algebra

Definition 10.18. Let V be a vector space over $\mathbb{C}$ of dimension $\dim V = n$ (we can replace $\mathbb{C}$ by any field). Define the **dual vector space** V^* as the set of linear maps $\alpha : V \to \mathbb{C}$. This is a vector space over $\mathbb{C}$.

Proposition 10.19

If $v_1, \ldots, v_n$ is a basis of V, then there is a natural basis $f_1, \ldots, f_n$ of V^* defined by the condition
$$f_i(v_j) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases},$$
the **Kronecker delta**. This basis $\{f_1, \ldots, f_n\}$ is called the dual basis. In particular, $\dim V^* = \dim V$.

Definition 10.20. A **bilinear form** $V \times V \to \mathbb{C}$ is a function (u, v) which is linear in each argument.

Definition 10.21. A bilinear form $(\ ,\ ) : V \times V \to \mathbb{C}$ is **symmetric** if $(u, v) = (v, u)$ for all $u, v \in V$.

Definition 10.22. Let $W \subseteq V$ be a vector subspace. Then
$$W^\perp = \{v \in V \mid (v, x) = 0 \text{ for all } x \in W\}.$$

Proposition 10.23. The set $W^\perp$ is a vector subspace.

Definition 10.24. A bilinear form $(\ ,\ ) : V \times V \to \mathbb{C}$ is called **non-degenerate** if $V^\perp = 0$.

If we choose a basis $v_1, \ldots, v_n$ of V, then $(\ ,\ )$ is given by a $n \times n$ matrix $A = (a_{ij})$, where $a_{ij} = (v_i, v_j)$ for all $1 \leq i, j \leq n$. Then for $v = \sum \lambda_i v_i$, $w = \sum \mu_j v_j$, we have
$$(v, w) = \sum a_{ij} \lambda_i \mu_j.$$

Exercise. A bilinear form $(\ ,\ )$ is non-degenerate if and only if A is invertible.

Proposition 10.25 (8.7)**.** Let $(\ ,\ )$ be a non-degenerate bilinear form on V. For $u \in V$, define $f_u \in V^*$ by the rule
$$f_u(x) = (u, x) \text{ for all } x \in V.$$
Then the map $V \to V^*$ given by $u \mapsto f_u$ is an **isomorphism of vector spaces**.

Proof. Linearity follows from the linearity of $(\ ,\ )$ in the first argument. (That $f_u \in V^*$ follows from the linearity of $(\ ,\ )$ in the second argument.)

Since $\dim V^* = \dim V$, it is enough to show that the kernel is 0. But the kernel of this map is $V^\perp$, and this is 0 since $(\ ,\ )$ is non-degenerate. $\square$

Proposition 10.26 (8.8)**.** Let $(\ ,\ )$ be a non-degenerate bilinear form on V. Let $W \subseteq V$ be a vector subspace. Then

1. $\dim W + \dim W^\perp = \dim V$,
2. if $W \cap W^\perp = \{0\}$, then $V = W \oplus W^\perp$.

Proof. For (1), choose a basis $v_1, \dots, v_r$ of W, and then extend it to a basis $v_1, \dots, v_n$ of V. By definition, $u \in W^\perp$ means that $f_u(x) = (u, x) = 0$ for all $x \in W$, which is equivalent to $f_u(v_i) = 0$ for $i = 1, \dots, r$. This happens if and only if $f_u \in \operatorname{Span}(f_{r+1}, \dots, f_n)$ (where we recall that $f_i(v_j) = \delta_{ij}$). Hence the image of $W^\perp$ in V^* has dimension $n - r$. But by Proposition 8.7, the map $V \to V^*$ that sends u to f_u is an isomorphism. Thus

$$\dim W^\perp = n - r = \dim V - \dim W.$$

This proves (1).

If we have subspaces $V_1 \subseteq V$ and $V_2 \subseteq V$ such that $\dim V_1 + \dim V_2 = \dim V$ and $V_1 \cap V_2 = \{0\}$, then $V = V_1 \oplus V_2$. We apply this to $V_1 = W$ and $V_2 = W^\perp$ to get (2). □

Definition 10.27. If $W \cap W^\perp = \{0\}$, let us call W a **non-degenerate** subspace of V.

10.5 Testing for semisimplicity

Lemma 10.28

Let I be an ideal of L.

1. For $x, y \in I$, $K_I(x, y) = K(x, y)$.
2. Define
$$I^\perp = \{x \in L : K(x, i) = 0 \text{ for all } i \in I\}.$$
Then $I^\perp$ is an ideal of L.

Proof. We prove each statement in turn.

1. Let B be a basis of I, and $x, y \in I$. Let
$$[\operatorname{ad} x]_B = M_x, \quad [\operatorname{ad} y]_B = M_y.$$
So $K_I(x, y) = \operatorname{Tr}(M_x M_y)$ by definition. Extend B to a basis B' of L. As I is an ideal, $\operatorname{ad} x$ maps $L \to I$, so
$$[\operatorname{ad} x]_{B'} = \begin{pmatrix} M_x & N_x \\ 0 & 0 \end{pmatrix}, \quad [\operatorname{ad} y]_{B'} = \begin{pmatrix} M_y & N_y \\ 0 & 0 \end{pmatrix}.$$
So
$$[(\operatorname{ad} x)(\operatorname{ad} y)]_{B'} = \begin{pmatrix} M_x M_y & M_x N_y \\ 0 & 0 \end{pmatrix}$$
and hence
$$K(x, y) = \operatorname{Tr}((\operatorname{ad} x)(\operatorname{ad} y)) = \operatorname{Tr}(M_x M_y) = K_I(x, y).$$
2. Let $x \in I^\perp$. Then
$$0 = K(x, i) = \operatorname{Tr}((\operatorname{ad} x)(\operatorname{ad} i)) \quad \text{for all } i \in I.$$
Let $y \in L, i \in I$. Then
$$\begin{aligned} K([xy], i) &= \operatorname{Tr}(\operatorname{ad}[xy] \operatorname{ad} i) \\ &= \operatorname{Tr}([\operatorname{ad} x, \operatorname{ad} y] \operatorname{ad} i) \\ &= \operatorname{Tr}(\operatorname{ad} x [\operatorname{ad} y, \operatorname{ad} i]) \quad \text{by Lemma 8.5} \\ &= \operatorname{Tr}(\operatorname{ad} x \operatorname{ad}[yi]) \\ &= K(x, [yi]) = 0 \quad \text{since } [yi] \in I. \end{aligned}$$
Hence $[xy] \in I^\perp$ for all $x \in I^\perp, y \in L$. So $I^\perp$ is an ideal.

$\square$

Theorem 10.29 (Cartan's second criterion)
A finite dimensional Lie algbera L over $\mathbb{C}$ is semisimple if and only if its Killing form is non-degenerate.

Proof. We prove each direction in turn.

- Proof of ($\Rightarrow$).
  Suppose L is semisimple. By a previous lemma, $L^\perp$ is an ideal of L. Also,
  $$K(x, y) = 0 \quad \text{for all } x, y \in L^\perp \text{ (by definition).}$$
  This implies that $L^\perp$ is solvable by Cartan's first criterion. Since L is semisimple i.e. $\mathrm{Rad}(L) = 0$, it follows $L^\perp = 0$, which means that K is non-degenerate.

- Proof of ($\Leftarrow$).
  we show that if L is non-semisimple, then K is degenerate (i.e. $L^\perp \neq 0$). Suppose $R = \mathrm{Rad}(L) \neq 0$. Let the derived series of R be
  $$R \supseteq R^{(1)} \supseteq R^{(2)} \supseteq \cdots \supseteq R^{(t)} = 0.$$
  Then $A = R^{(t-1)}$ is a nonzero abelian ideal of L. We will show that $A \subseteq L^\perp$.

  We claim that the map $(\operatorname{ad} a)(\operatorname{ad} x) : L \to L$ is nilpotent for any $a \in A, x \in L$. For $l \in L$, the composition $(\operatorname{ad} a)(\operatorname{ad} x)(\operatorname{ad} a)$ sends
  $$l \mapsto [a[l]] \mapsto [x[a[l]]] \mapsto 0$$
  as A is abelian. Therefore, $(\operatorname{ad} a)(\operatorname{ad} x)(\operatorname{ad} a) = 0$, so $((\operatorname{ad} a)(\operatorname{ad} x))^2 = 0$.

  Therefore, $\mathrm{Tr}((\operatorname{ad} a)(\operatorname{ad} x)) = 0$. Hence
  $$K(a, x) = \mathrm{Tr}((\operatorname{ad} a)(\operatorname{ad} x)) = 0 \quad \text{for all } a \in A, x \in L,$$
  which shows $A \subseteq L^\perp$, hence $L^\perp \neq 0$.

$\square$

Example 10.30
The Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ is semisimple (because it is simple), and the matrix of its Killing form with respect to the basis e, h, f is
$$\begin{pmatrix} 0 & 4 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & 8 \end{pmatrix}.$$

10.6 Structure of semisimple Lie algebras

Lemma 10.31. Let L be a semisimple Lie algebra over $\mathbb{C}$. Suppose I is a non-zero ideal. Then

$$L = I \oplus I^{\perp} \quad \text{(perpendicular with respect to } K\text{)}$$

and I is itself semisimple.

Proof. Since $K(x, y) = 0$ for any $x, y \in I \cap I^{\perp}$ by definition of $I^{\perp}$, Cartan's first criterion shows that $I \cap I^{\perp}$ is solvable. As L is semisimple, $I \cap I^{\perp} = 0$. Hence, as K is non-degenerate by a previous theorem, $L = I \oplus I^{\perp}$ by a previous proposition. As $I \cap I^{\perp} = 0$, the restriction of K to I is non-degenerate. Hence so is K_I, the Killing form of I, by a previous lemma. Therefore, I is semisimple by a previous theorem. □

Theorem 10.32

Let L be a finite dimensional Lie algbera over $\mathbb{C}$. We have that L is semisimple if and only if there exists simple ideals $L_1, \dots, L_r$ of L such that $L = \bigoplus_{i=1}^{r} L_i$.

Remark 10.33. This means

1. $L = L_1 \oplus \cdots \oplus L_r$, as vector spaces,
2. $[L_i, L_j] \subseteq L_i \cap L_j = 0$ for all $i \neq j$.

Furthermore, for any finite dimensional Lie algebra L over $\mathbb{C}$, this means

- $\operatorname{Rad}(L)$ is the maximal solvable ideal of L,
- $L/\operatorname{Rad}(L)$ is semisimple, so it is the direct sum of simple ideals.

Proof. We prove each direction in turn.

- Proof of ($\Rightarrow$).
  We first proceed by induction on $\dim L$ to show that if L is semisimple, then L is a direct sum of simple ideals. The statement is trivial when L is simple (in particular, when $\dim L = 1$). So let $0 \neq I \subseteq L$ be an ideal. Then by a previous lemma, $L = I \oplus I^{\perp}$ and the ideals $I, I^{\perp}$ are semisimple. By the inductive hypothesis:

  $$I = L_1 \oplus \cdots \oplus L_m, \quad I^{\perp} = M_1 \oplus \cdots \oplus M_n$$

  where L_i, M_i are simple ideals of $I, I^{\perp}$, respectively. Then

  $$[L_i, L_j] = [L_i, I + I^{\perp}] = [L_i, I] + [L_i, I^{\perp}] = [L_i, I] \subseteq L_i,$$

  since $[L_i, I^{\perp}] \subseteq [I, I^{\perp}] \subseteq I \cap I^{\perp} = 0$. Hence all L_i are ideals of L, and similarly, M_i are ideals of L. Hence

  $$L = L_1 \oplus \cdots \oplus L_m \oplus M_1 \oplus \cdots \oplus M_n$$

  is a direct sum of simple ideals.

- Proof of ($\Leftarrow$).
  let $L = L_1 \oplus \cdots \oplus L_r$, where L_i are simple ideals. Let I be a soluble ideal of L. Then
  $$[IL_i] \subseteq I \cap L_i,$$
  so $[I, L_i] = 0$, since L_i is simple and I is soluble. Hence
  $$[I, L] = [I, L_1] \oplus \cdots \oplus [I, L_r] = 0$$
  and $I \subseteq Z(L) = \bigoplus_i Z(L_i) = 0$.

□

10.7 Derivations of semisimple Lie algebras

Note 10.34. We apply Cartan's second criterion to show that the only derivations of a complex semisimple Lie algebra are those of the form $\mathrm{ad}(x)$ for $x \in L$.

Proposition 10.35. The set $\mathrm{ad}\, L = \{\mathrm{ad}\, x : x \in L\}$ is an ideal of $\mathrm{Der}\, L$.

Proof. Let $\delta \in \mathrm{Der}\, L$ and $x, y \in L$. Then

$$[\delta, \mathrm{ad}\, x](y) = \delta([x, y]) - [x, \delta(y)] = [\delta(x), y] + [x, \delta(y)] - [x, \delta(y)] = [\delta(x), y] = (\mathrm{ad}(\delta(x)))(y).$$

Hence
$$[\delta, \mathrm{ad}\, x] = \mathrm{ad}(\delta(x)).$$
So $\mathrm{ad}\, L$ is an ideal. □

Proposition 10.36

If L is a semisimple Lie algebra over $\mathbb{C}$, then $\mathrm{ad}\, L = \mathrm{Der}\, L$.

Proof. Let $M = \mathrm{ad}\, L$. As $Z(L) = 0$, $\mathrm{ad} : L \to M$ is an isomorphism, so M is also semisimple, and M is an ideal of $\mathrm{Der}\, L$. Let K be the Killing form of $\mathrm{Der}\, L$. We claim that $M^{\perp} = 0$. By Lemma 8.10, the Killing form K_M of M is the restriction of K to M. By Theorem 8.11, K_M is non-degenerate, so $M \cap M^{\perp} = 0$. Note that
$$[M, M^{\perp}] \subseteq M \cap M^{\perp} = 0,$$
and hence for $\delta \in M^{\perp}$ and $\mathrm{ad}\, x \in M$, we have $[\delta, \mathrm{ad}\, x] = 0$. By the proof of Proposition 9.4,
$$\mathrm{ad}\, \delta(x) = [\delta, \mathrm{ad}\, x] = 0,$$
so $\delta(x) = 0$ for all $x \in L$, i.e. $\delta = 0$. Hence we have shown that $M^{\perp} = 0$. This implies that
$$(\mathrm{Der}\, L)^{\perp} \subseteq M^{\perp} = 0,$$
and therefore K is non-degenerate, so by Proposition 8.8,
$$\dim(\mathrm{Der}\, L) = \dim M + \dim M^{\perp} = \dim M,$$
as $\dim M^{\perp} = 0$, which implies that $M = \mathrm{Der}\, L$. □

10.8 Abstract Jordan decomposition

Note 10.37. Given a representation $\varphi : L \to \mathfrak{gl}(V)$ of a Lie algebra L, we may consider the Jordan decomposition of the linear maps $\varphi(x)$ for $x \in L$.

For a general Lie algebra there is not much that can be said about this decomposition without knowing more about the representation φ. For example, if L is the 1-dimensional abelian Lie algebra, spanned, say by x, then we may define a representation of L on a vector space V by mapping x to any element of $\mathfrak{gl}(V)$. So the Jordan decomposition of $\varphi(x)$ is essentially arbitrary.

Theorem 10.38

Let L be a finite-dimensional semisimple Lie algebra over $\mathbb{C}$. Then every $x \in L$ can be expressed uniquely as

$$x = d + n$$

where

1. $d, n \in L$,
2. $\operatorname{ad} d : L \to L$ is diagonalizable, and $\operatorname{ad} n : L \to L$ is nilpotent,
3. $[d, n] = 0$.

Moreover, for $y \in L$,

$$[x, y] = 0 \implies [d, y] = [n, y] = 0.$$

Definition 10.39. We call $x = d + n$ the **Jordan decomposition** of x. We call d a semisimple element of L and n a nilpotent element.

Definition 10.40. So an element $x \in L$ is called **semisimple** if $\operatorname{ad} x : L \to L$ is diagonalisable.

Proposition 10.41. Let $L \subseteq \mathfrak{gl}(V)$ be a semisimple Lie subalgebra over $\mathbb{C}$. Then the Jordan decomposition

$$x = d + n$$

in Theorem 9.1 is the same as the Jordan decomposition of x as a linear map $V \to V$.

Proof. Let

$$x = d' + n'$$

be the Jordan decomposition of $x : V \to V$. So $d' : V \to V$ is diagonalizable, $n' : V \to V$ is nilpotent, and $[d', n'] = n'd' - d'n'$. By Lemma 8.2,

$$\operatorname{ad}(x) = \operatorname{ad}(d') + \operatorname{ad}(n')$$

is the Jordan decomposition of $\operatorname{ad} x : L \to L$. If $x = d + n$ as in the theorem above, then by (1)–(3)

$$\operatorname{ad}(x) = \operatorname{ad}(d) + \operatorname{ad}(n)$$

is also the Jordan decomposition of $\operatorname{ad} x : L \to L$. By uniqueness of Jordan decomposition, $\operatorname{ad}(d') = \operatorname{ad}(d)$ and $\operatorname{ad}(n') = \operatorname{ad}(n)$.

As L is semisimple, $Z(L) = 0$, so $\operatorname{ad} : L \to \mathfrak{gl}(L)$ is injective. Therefore, $d' = d$ and $n' = n$. ∎

Proposition 10.42

Let L be a Lie algebra over $\mathbb{C}$. Let $\delta \in \operatorname{Der} L$ have Jordan decomposition (as a linear map $L \to L$)

$$\delta = \sigma + \nu$$

where $\sigma : L \to L$ is diagonal, $\nu : L \to L$ is nilpotent, and $\sigma\nu = \nu\sigma$. Then $\sigma, \nu \in \operatorname{Der} L$.

Proof. Let $\lambda_1, \dots, \lambda_r$ be distinct eigenvalues of $\sigma : L \to L$, and m_i be the size of the largest λ_i-Jordan block. Then

$$L = \bigoplus_{i=1}^{r} L_{\lambda_i}$$

where

$$L_{\lambda_i} = \ker(\sigma - \lambda_i I)^{m_i}.$$

On each L_{λ_i}, σ acts as $\lambda_i I$, and ν as a strictly upper-triangular matrix. For each $\lambda \in \mathbb{C}$, define

$$L_\lambda = \{x \in L : (\sigma - \lambda I)^k(x) = 0 \text{ for some } k\} = \begin{cases} L_{\lambda_i}, & \text{if } \lambda = \lambda_i, \\ 0, & \text{if } \lambda \neq \lambda_i. \end{cases}$$

We claim that

$$[L_\lambda, L_\mu] \subseteq L_{\lambda+\mu}.$$

We show that for any $n \in \mathbb{N}$,

$$(\delta - (\lambda+\mu)I)^n[xy] = \sum_{k=0}^{n} \binom{n}{k} [(\sigma - \lambda I)^k(x), (\sigma - \mu I)^{n-k}(y)]$$

by induction on n. The base case $n = 1$ is clear:

$$\text{RHS} = [x, (\delta-\mu)(y)] + [(\delta-\lambda)(x), y] = [x, \delta(y)] + [\delta(x), y] - (\lambda+\mu)[x, y],$$

$$= \delta([x, y]) - (\lambda+\mu)[x, y] = (\delta - (\lambda+\mu)I)[xy] = \text{LHS}.$$

The inductive step is omitted. By equation (1), for $x \in L_\lambda, y \in L_\mu$,

$$(\delta - (\lambda+\mu)I)^n[xy] = 0$$

for sufficiently large n. Hence $[xy] \in L_{\lambda+\mu}$, which shows that $[L_\lambda, L_\mu] \subseteq L_{\lambda+\mu}$. Now, we use this to show that $\sigma \in \operatorname{Der} L$. Recall that L_λ is a λ-eigenspace for σ. For $x \in L_\lambda, y \in L_\mu$, $[xy] \in L_{\lambda+\mu}$, so

$$\sigma(x) = \lambda x, \quad \sigma(y) = \mu y \quad \text{and} \quad \sigma[xy] = (\lambda+\mu)[xy].$$

Hence

$$[\sigma(x), y] + [x, \sigma(y)] = \lambda y + \mu x + [xy] = (\lambda+\mu)[xy] = \sigma[xy],$$

showing that $\sigma \in \operatorname{Der} L$. Hence $\nu = \delta - \sigma \in \operatorname{Der} L$. $\square$

Proof of Theorem 10.38. Let L be a semisimple Lie algebra over $\mathbb{C}$ and let $x \in L$. Then $x \in \operatorname{Der} L$. Let the Jordan decomposition of $\operatorname{ad} x : L \to L$ be

$$\operatorname{ad} x = \sigma + \nu$$

(for σ diagonalizable, ν nilpotent, $\sigma\nu = \nu\sigma$). By Proposition 9.6, $\sigma, \nu \in \operatorname{Der} L$. By Proposition 9.5, $\operatorname{Der} L = \operatorname{ad} L$, so there exist $d, n \in L$ such that

$$\sigma = \operatorname{ad}(d), \quad \nu = \operatorname{ad}(n).$$

Hence

$$\operatorname{ad} x = \operatorname{ad}(d) + \operatorname{ad}(n) = \operatorname{ad}(d+n).$$

As $Z(L) = 0$, $\operatorname{ad} : L \to \operatorname{ad} L = \operatorname{Der} L$ is an isomorphism, so

$$x = d + n.$$

As $d, n \in L$, part (1) of Theorem 9.1 holds. As $\sigma = \operatorname{ad} d$, $\nu = \operatorname{ad} n$, part (2) holds. Also

$$\operatorname{ad}[dn] = [\operatorname{ad}(d), \operatorname{ad}(n)] = [\sigma, \nu] = 0,$$

hence $[dn] = 0$, so part (3) holds. As σ and ν are unique (by uniqueness of Jordan decomposition), so are d and n. We finally prove the last part of Theorem 9.1. Let $y \in L$ with $[xy] = 0$. By Lemma 8.1, σ and $\nu = \sigma - \operatorname{ad} x$ are polynomials in $\operatorname{ad} x$. Say

$$\nu = c_r(\operatorname{ad} x)^r + \cdots + c_1 \operatorname{ad} x + c_0 I.$$

As $(\operatorname{ad}(x))(y) = [xy] = 0$, this means

$$\nu(y) = c_0 y.$$

Since ν is nilpotent, $c_0 = 0$, so $\nu(y) = 0$. Hence

$$0 = \nu(y) = (\operatorname{ad} n)(y) = [ny],$$

so $[ny] = 0$, and then $[dy] = 0$ as $[xy] = 0$. □

11 Root space decomposition

Note 11.1. We now work towards the classification of the complex simple Lie algebras.

We have seen that $\mathfrak{sl}(2, \mathbb{C})$ is a simple Lie algebra, with basis e, f, h such that

- ad h is diagonalisable — h is a semisimple element
- e, f, h are a basis of eigenvectors for ad h.

For $\mathfrak{sl}(3, \mathbb{C})$ replace h by

$$H = \{\text{diagonal matrices}\} = \left\{ \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} : \sum a_i = 0 \right\}$$

a 2-dimensional abelian subalgebra.

Lemma 11.2

We have that $(\text{ad } h)(e_{ij}) = [h, e_{ij}] = (a_i - a_j)e_{ij}$. Therefore,

$$\mathfrak{sl}(3, \mathbb{C}) = H \oplus \bigoplus_{i,j} \text{span}(e_{ij}),$$

a direct sum of weight spaces for ad H. The weights corresponding to each of these weight spaces are

- For the space H the weight is 0.
- For the space $\text{span}(e_{ij})$ the weights are $\varepsilon_i - \varepsilon_j$ where $\varepsilon_i : H \to \mathbb{C}$ such that $h \mapsto a_i$.

Our strategy is therefore:

1. to find an abelian Lie subalgebra H of L that consists entirely of semisimple elements; and
2. to decompose L into weight spaces for the action of ad H
3. and then exploit this decomposition to determine information about the structure constants of L.

11.1 Preliminary results

Remark 11.3. In this section, we assume L is a Lie algebra over $\mathbb{C}$, and H is an abelian subalgebra of L consisting of semisimple elements.

Lemma 11.4

Let L be a Lie algebra over $\mathbb{C}$, and H is an abelian subalgebra of L consisting of semisimple elements. Then L has a basis of common eigenvectors for all ad h, for all $h \in H$.

Note 11.5. There is a single basis for L such that each basis element is an eigenvector of every ad h for all $h \in H$.

Proof. Let $h_1, \dots, h_r$ be a basis for H and $\alpha_i = \text{ad } h_i : L \to L$. Then

$$[\alpha_i, \alpha_j] = \text{ad } [h_i, h_j] = 0$$

since H is abelian. So $\alpha_1, \dots, \alpha_r$ are commuting, diagonalizable linear maps $L \to L$. Such maps have a basis of common eigenvectors. □

Definition 11.6. Let $x \in L$ be a common eigenvector for ad H. Define **weight** $\alpha : H \to \mathbb{C}$ (i.e. $\alpha \in H^*$) by

$$(\text{ad}\, h)(x) = [h, x] = \alpha(h)x$$

for all $h \in H$.

Definition 11.7. The **weight space** of α is

$$L_\alpha = \{x \in L : [h, x] = \alpha(h)x \text{ for all } h \in H\}$$

.

Example 11.8

One of these space is the 0**-weight space**:

$$L_0 = \{x \in L : [h, x] = 0 \text{ for all } h \in H\}.$$

Note that $H \subseteq L_0$.

Proposition 11.9. Let Φ denote the set of non-zero $\alpha \in H^*$ for which $L_\alpha \neq 0$. Then we can write

$$L = L_0 \oplus \bigoplus_{\alpha \in \Phi} L_\alpha.$$

Since L is finite-dimensional, this implies that Φ is finite.

Proposition 11.10

Let $\alpha, \beta \in H^*$.

1. $[L_\alpha, L_\beta] \subseteq L_{\alpha+\beta}$
2. If $\alpha + \beta \neq 0$, then $K(L_\alpha, L_\beta) = 0$.
3. Suppose L is semisimple. Then the restriction of K to L_0 is non-degenerate (i.e. $L_0 \cap L_0^\perp = 0$).

Proof. We prove each statement in turn.

1. Let $x \in L_\alpha$, $y \in L_\beta$. For $h \in H$,

   $$[h, [x, y]] = [[h, x], y] + [x, [h, y]] \quad \text{by the Jacobi identity.}$$

   Thus,

   $$= \alpha(h)[xy] + \beta(h)[xy] = (\alpha + \beta)(h)[xy],$$

   and hence $[xy] \in L_{\alpha+\beta}$.

2. Let $h \in H$ with $(\alpha + \beta)(h) \neq 0$. For $x \in L_\alpha$, $y \in L_\beta$,

   $$\alpha(h)K(x, y) = K([hx], y) \quad \text{as } K \text{ is bilinear,}$$

   $$= -K(x, [hy]) \quad \text{as } K \text{ is bilinear,}$$

   $$= -K(x, \beta(h)y) \quad \text{by Lemma 8.5,}$$

   $$= -\beta(h)K(x, y) \quad \text{as } K \text{ is bilinear.}$$

   Hence $(\alpha + \beta)(h)K(x, y) = 0$, so $K(x, y) = 0$.

3. Let $y \in L_0 \cap L_0^\perp$. Then $K(L_0, y) = 0$. For $x \in L$, use the weight space decomposition (1) to write
$$x = x_0 + \sum_{\alpha \in \Phi} x_\alpha \quad (x_\alpha \in L_\alpha).$$
By (2), $K(L_0, L_\alpha) = 0$ if $\alpha \neq 0$. So $K(x_\alpha, y) = 0$ for all $\alpha \in \Phi$. Also, $K(x_0, y) = 0$ by assumption. Hence
$$K(x, y) = 0 \quad \text{for all } x \in L.$$
So $y \in L^\perp$. As L is semisimple, K is non-degenerate, so $L^\perp = 0$. Hence $y = 0$.

□

Corollary 11.11

If $x \in L_\alpha$, where $\alpha \neq 0$, $\alpha \in H^*$, then ad x is nilpotent.

Proof. For any weight $\beta \in \Phi \cup \{0\}$, we have
$$\begin{aligned}(\text{ad } x)(L_\beta) &\subseteq L_{\alpha+\beta},\\ (\text{ad } x)^2(L_\beta) &\subseteq L_{2\alpha+\beta},\\ &\vdots\\ (\text{ad } x)^r(L_\beta) &\subseteq L_{r\alpha+\beta},\end{aligned}$$
by a previous proposition. But Φ is finite, so for some r, we will have $r\alpha + \beta \notin \Phi$. This means that $(\text{ad } x)^r(L_\beta) = 0$.

Once again, since Φ is finite and for each $\beta \in \Phi \cup \{0\}$, we can find r such that $(\text{ad } x)^r(L_\beta) = 0$, we can take the maximum of these r to obtain one r such that
$$(\text{ad } x)^r(L_\beta) = 0$$
for any $\beta \in \Phi \cup \{0\}$. Since
$$L = L_0 \oplus \bigoplus_{\alpha \in \Phi} L_\alpha$$
we see that $(\text{ad } x)^r$ is the zero transformation $L \to L$. □

11.2 Cartan subalgebras

Definition 11.12. A Lie subalgebra H of a Lie algebra L is said to be a **Cartan subalgebra** if

1. H is abelian,
2. every element $h \in H$ is semisimple,
3. H is maximal among subalgebras $H \subseteq L$ satisfying the properties above.

Proposition 11.13. Let L be a finite dimensional semisimple Lie algebra over $\mathbb{C}$, then L has a Cartan subalgebra.

Proof. Recall that each $x \in L$ has a Jordan decomposition $x = s+n$, where s is semisimple and n is nilpotent, with $s, n \in L$ and $[s, n] = 0$.

Suppose that the semisimple part of every $x \in L$ is zero. Then all elements of L are nilpotent. Hence $\operatorname{ad} x$ is nilpotent for any $x \in L$. By Engel's Theorem, this implies that L is a nilpotent Lie algebra — a contradiction, since L is semisimple by assumption.

Therefore, there exists some $x \in L$ whose semisimple part is nonzero. Hence semisimple elements exist. Let $s \in L$ be such a semisimple element. Then $\operatorname{Span}(s)$ is an abelian subalgebra of L consisting of semisimple elements.

Now, we can extend $\operatorname{Span}(s)$ to a maximal abelian subalgebra $H \subseteq L$ consisting entirely of semisimple elements. Such an H satisfies the defining conditions of a Cartan subalgebra: it is nilpotent (in this case, abelian) and self-normalising. Hence, a Cartan subalgebra exists. □

Definition 11.14. For a subset $X \subseteq L$, we define the **centraliser** of X in L by

$$C_L(X) = \{l \in L : [l, x] = 0 \text{ for all } x \in X\}.$$

Proposition 11.15. $C_L(X)$ is a Lie subalgebra of L.

Lemma 11.16. Suppose H is a Lie subalgebra of L such that:

1. H consists of semisimple elements;
2. $C_L(H) = H$.

Then H is a Cartan subalgebra in L.

Proof. Note first that $H \subseteq C_L(H)$ always holds. If equality occurs and H is made entirely of semisimple elements, then it is abelian and maximal among such subalgebras.

Suppose not, i.e., there exists a larger subalgebra $H_1 \supsetneq H$ with the same two properties. Then we would have $H_1 \subseteq C_L(H) = H$, a contradiction. Thus H is maximal among abelian subalgebras made of semisimple elements and hence is a Cartan subalgebra. □

Example 11.17

Let $L = \mathfrak{sl}(n, \mathbb{C}) = \{n \times n \text{ matrices with trace } 0\}$, and let

$$H = \{\text{diagonal } n \times n \text{ matrices with trace } 0\}.$$

Then $H \subseteq L$ is clearly abelian and consists of semisimple (diagonalisable) elements. Also,

$$\mathfrak{sl}(n, \mathbb{C}) = H \oplus \bigoplus_{\substack{1 \le i,j \le n \\ i \ne j}} \mathbb{C}E_{ij},$$

so $C_L(H) = H$, and thus H is a Cartan subalgebra.

Theorem 11.18

Let L be a semisimple Lie algebra over $\mathbb{C}$. Let H be a Cartan subalgebra in L. Then

$$C_L(H) = H.$$

Proof. Let L be semisimple with Cartan subalgebra H.

- **Step 1.** Choose $h \in H$ such that $\dim C_L(h)$ is minimal. We aim to show $C_L(h) = C_L(H)$. Suppose not, so $C_L(h) \neq C_L(H)$. Then there exists $s \in H$ such that
$$C_L(h) \cap C_L(s) \subsetneq C_L(h).$$
Let us build a basis for these centralisers to analyse the structure. Choose a basis $c_1, \ldots, c_n$ of $C_L(h) \cap C_L(s)$. Extend this to a basis
$$c_1, \ldots, c_n, x_1, \ldots, x_p$$
of $C_L(h)$ consisting of eigenvectors for $\mathrm{ad}(s)$, and
$$c_1, \ldots, c_n, y_1, \ldots, y_q$$
of $C_L(s)$, eigenvectors for $\mathrm{ad}(h)$. Then extend to
$$c_1, \ldots, c_n, x_1, \ldots, x_p, y_1, \ldots, y_q, z_1, \ldots, z_r$$
a basis of L of simultaneous eigenvectors for $\mathrm{ad}(h), \mathrm{ad}(s)$.

  Now, note that for each i,
$$[sx_i] \neq 0, \quad [hy_j] \neq 0, \quad [hz_i] = \alpha_i z_i, \quad [sz_i] = \beta_i z_i,$$
where $\alpha_i, \beta_i \neq 0$. Consider
$$[h + \lambda s, z_i] = (\alpha_i + \lambda\beta_i) z_i.$$
Choosing λ such that $\alpha_i + \lambda\beta_i \neq 0$, we get $z_i \notin C_L(h + \lambda s)$. So,
$$C_L(h + \lambda s) \subsetneq C_L(h),$$
contradicting the minimality. Hence,
$$C_L(h) = C_L(H).$$

- **Step 2.** Show $C_L(h)$ is nilpotent. Let $x \in C_L(h)$, with Jordan decomposition $x = d + n$, then
$$[h, x] = 0 \Rightarrow [h, d] = [h, n] = 0.$$
Since $d \in H$, the restriction $\mathrm{ad}(d) : C_L(h) \to C_L(h)$ is zero. Then $\mathrm{ad}(x) = \mathrm{ad}(n)$, which is nilpotent. Hence $\mathrm{ad}(C_L(h))$ consists of nilpotent maps, so $C_L(h)$ is nilpotent (Engel's theorem).

- **Step 3.** Show $C_L(h) \subseteq H$.

  By Step 2, $C_L(h)$ is nilpotent. Hence, $\mathrm{ad}(x)$ is strictly upper triangular for $x \in C_L(h)$. Write $x = d + n$ (Jordan form), and since $K(n, y) = 0$ for all $y \in C_L(h)$, by the non-degeneracy of the Killing form K on $C_L(h)$, we get $n = 0$. Hence $x = d \in H$.

So,
$$C_L(h) \subseteq H.$$
By Steps 1 and 3, $C_L(H) = H$, as required. $\square$

11.3 Definition of the root space decomposition

Definition 11.19. Let L be a semisimple Lie algebra over $\mathbb{C}$ and H be a Cartan subalgebra. We have the **root space decomposition**

$$L = H \oplus \bigoplus_{\alpha \in \Phi} L_\alpha$$

where

$$L_\alpha = \{x \in L : [h, x] = \alpha(h)x \quad \text{for all } h \in H\}$$

and $\Phi \subset H^* \setminus \{0\}$, the set of **roots** of L (with respect to H).

Note 11.20. That is, $\alpha \in \Phi$ are **roots** and L_α is the associated **root space**.

11.4 Subalgebras isomorphic $\mathfrak{sl}(2)$-subalgebras

Proposition 11.21

Let $\alpha \in \Phi$. Then:

1. $-\alpha \in \Phi$,
2. Let $0 \neq x \in L_\alpha$. Then there exists $y \in L_{-\alpha}$ such that

$$\mathfrak{s} = \mathrm{Span}(x, y, [x, y])$$

is a subalgebra of L isomorphic to $\mathfrak{sl}(2, \mathbb{C})$.

Proof. We prove each statement in turn.

1. Let $0 \neq x \in L_\alpha$. Since the Killing form K is non-degenerate, there exists $y \in L$ such that $K(x, y) \neq 0$. Writing

$$y = y_0 + \sum_{\beta \in \Phi} y_\beta \quad (y_0 \in H,\ y_\beta \in L_\beta),$$

and using the orthogonality properties of root spaces under the Killing form, we find that $K(x, y) = K(x, y_{-\alpha}) \neq 0$. Hence $y_{-\alpha} \neq 0$, and so $-\alpha \in \Phi$.

2. Let $x \in L_\alpha$, $y \in L_{-\alpha}$ with $K(x, y) \neq 0$. Then the element $[x, y] \in H$, and we claim that it's non-zero. Indeed, using Lemma 8.5 and the non-degeneracy of K,

$$K([x, y], u) = \alpha(u)K(x, y) \neq 0$$

for some $u \in H$, so $[x, y] \neq 0$. Define

$$\mathfrak{s} = \mathrm{Span}(x, y, [x, y]) \subset L.$$

This is closed under Lie brackets and so forms a subalgebra. We show $\mathfrak{s} \cong \mathfrak{sl}(2, \mathbb{C})$.

□

Proposition 11.22. Let $h = [x, y] \in H$. Then

$$[[x, y], x] = \alpha([x, y])x, \quad [[x, y], y] = -\alpha([x, y])y.$$

So if we rescale:

$$e_\alpha = x, \quad e_{-\alpha} = y, \quad h_\alpha = [x, y],$$

and define

$$[e_\alpha, e_{-\alpha}] = h_\alpha, \quad [h_\alpha, e_\alpha] = 2e_\alpha, \quad [h_\alpha, e_{-\alpha}] = -2e_{-\alpha},$$

then these relations satisfy the $\mathfrak{sl}(2, \mathbb{C})$ Lie algebra structure. Hence,

$$\mathfrak{s} = \mathfrak{sl}(\alpha) \cong \mathfrak{sl}(2, \mathbb{C}).$$

Example 11.23

Take $L = \mathfrak{sl}(3, \mathbb{C})$. We have the root space decomposition:

$$L = H \oplus \bigoplus_{i \neq j} \operatorname{Span}(E_{ij})$$

with roots $\epsilon_i - \epsilon_j$, where

$$\epsilon_i : \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} \mapsto a_i.$$

For a fixed root $\alpha = \epsilon_i - \epsilon_j$, the subalgebra

$$\mathfrak{sl}(\alpha) = \operatorname{Span}(E_{ij}, E_{ji}, h_{ij}), \quad \text{where } h_{ij} = E_{ii} - E_{jj},$$

is isomorphic to $\mathfrak{sl}(2, \mathbb{C})$.

> **Note 11.24.** Each root α of $\mathfrak{sl}(3, \mathbb{C})$ defines a copy of $\mathfrak{sl}(2, \mathbb{C})$ sitting inside the full algebra. The pair of off-diagonal elements E_{ij}, E_{ji} act like the raising/lowering operators, and h_{ij} behaves like a diagonal Cartan element. These subalgebras give local symmetries analogous to those of $\mathfrak{sl}(2)$.

Module structure. We can regard L as an $\mathfrak{sl}(\alpha)$-module, with the action defined by:

$$s \cdot \ell = [s, \ell] \qquad (s \in \mathfrak{sl}(\alpha),\ \ell \in L).$$

Proposition 11.25

Let $\alpha \in \Phi$ and $\beta \in \Phi \cup \{0\}$. Define the subspace

$$M = \bigoplus_{\substack{c \in \mathbb{Z} \\ \beta + c\alpha \in \Phi}} L_{\beta + c\alpha}.$$

Then M is an $\mathfrak{sl}(\alpha)$-submodule of L.

Proof. By a previous proposition, we have:

$$[L_{\beta+c\alpha}, L_{\pm\alpha}] \subseteq L_{\beta+(c\pm1)\alpha}, \quad \text{and} \quad [L_{\beta+c\alpha}, H] \subseteq L_{\beta+c\alpha},$$

so the Lie bracket of $\mathfrak{sl}(\alpha)$ with M is closed within M, making it an $\mathfrak{sl}(\alpha)$-submodule. □ □

Definition 11.26. The set of roots

$$\{\beta + c\alpha : c \in \mathbb{Z}\} \cap \Phi$$

is called the α**-string through** β.

Example 11.27

Let $L = \mathfrak{sl}(3, \mathbb{C})$, $\alpha = \epsilon_2 - \epsilon_3$, $\beta = \epsilon_1 - \epsilon_2$. Then the α-string through β is

$$\beta, \quad \beta + \alpha = \epsilon_1 - \epsilon_3.$$

So the corresponding submodule is:

$$M = \mathrm{Span}(E_{12}, E_{13}) = \left\{ \begin{pmatrix} 0 & * & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}, \quad \mathfrak{sl}(\alpha) = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \right\}.$$

Proposition 11.28

Let $\alpha \in \Phi$. Then:

1. $\dim L_\alpha = 1$,
2. If $n \in \mathbb{Z} \setminus \{0\}$ and $n\alpha \in \Phi$, then $n = \pm 1$.

Proof. Define

$$W := \mathrm{Span}(e_{-\alpha}, H, L_{n\alpha} : n \in \mathbb{N}).$$

Then W is invariant under $\mathrm{ad}\, e_\alpha$, $\mathrm{ad}\, e_{-\alpha}$, and $\mathrm{ad}\, H$, making it an $\mathfrak{sl}(\alpha)$-submodule. Let $(\mathrm{ad}\, e_\alpha)_W$ be the restriction of $\mathrm{ad}\, e_\alpha$ to W. Then:

$$\mathrm{Tr}((\mathrm{ad}\, e_\alpha)_W \circ (\mathrm{ad}\, e_{-\alpha})_W) = 0,$$

and also

$$\mathrm{Tr}(\mathrm{ad}[e_\alpha, e_{-\alpha}])_W = \mathrm{Tr}(\mathrm{ad}\, h_\alpha)_W.$$

Now, we compute the trace of $\mathrm{ad}\, h_\alpha$ on W. Writing

$$(\mathrm{ad}\, h_\alpha)_W = \begin{pmatrix} -\alpha(h) & & \\ & 0I & \\ & & n\alpha(h)I \end{pmatrix},$$

Note 11.29. Where $0I$ represent a block matrix with 0 as the diagonal and $n\alpha(h)I$ is also anohter block matrix with $n\alpha(h)$ as the diagonal.

with contributions from $e_{-\alpha}$, H, and $L_{n\alpha}$ respectively. Taking the trace, we find:

$$0 = -\alpha(h) + \sum_{n\geq 1} n\alpha(h) \dim L_{n\alpha}.$$

Taking $\alpha(h) = 2$, we obtain:

$$\sum_{n\geq 1} n \dim L_{n\alpha} = 1.$$

Hence, $\dim L_\alpha = 1$ and $L_{n\alpha} = 0$ for $n > 1$. A similar argument applies to $L_{-\alpha}$, showing $\dim L_{-\alpha} = 1$ and $L_{-n\alpha} = 0$ for $n > 1$. $\square$

Proposition 11.30

Let $S = \mathrm{Span}(e, f, h) \cong \mathfrak{sl}(2, \mathbb{C})$, and let V be a finite-dimensional S-module. Then every eigenvalue of the linear map

$$v \mapsto hv \qquad (v \in V)$$

is an integer.

Proof. Recall that the irreducible S-modules are V_d, and on V_d, the element h has eigenvalues

$$d, d-2, \ldots, -d \in \mathbb{Z}.$$

Now take a composition series:

$$V = W_0 \supset W_1 \supset \cdots \supset W_r = 0,$$

where each W_i is a submodule and each quotient W_i/W_{i+1} is irreducible. Thus,

$$\frac{W_i}{W_{i+1}} \cong V_{d_i}$$

for some $d_i \in \mathbb{Z}_{\geq 0}$. With respect to a suitable basis of V, the action of h is block diagonal:

$$\begin{pmatrix} d_0 & & & & & & \\ & \ddots & & & & & \\ & & -d_0 & & & & \\ & \star & & d_1 & & & \\ & & & & \ddots & & \\ & & & & & -d_1 & \\ & & & & & & \ddots \end{pmatrix}$$

Each diagonal block corresponds to an irreducible module V_{d_i}, and since h acts diagonally with integer eigenvalues on each block, all eigenvalues of h on V are integers. $\square$

Proposition 11.31
Let $\alpha, \beta \in \Phi$ with $\beta \neq \pm\alpha$.

1. $\beta(h_\alpha) \in \mathbb{Z}$.
2. The α-root string through β is
$$\beta - r\alpha,\ \beta - (r-1)\alpha,\ \ldots,\ \beta,\ \ldots,\ \beta + q\alpha$$
where $r, q \geq 0$ and $\beta(h_\alpha) = r - q$.
3. If $\alpha + \beta \in \Phi$, then
$$[e_\alpha, e_\beta] = \lambda e_{\alpha+\beta} \quad (\lambda \neq 0).$$
4. $\beta - \beta(h_\alpha)\alpha \in \Phi$.

Proof. Let
$$M = \sum_{c \in \mathbb{Z}} L_{\beta + c\alpha},$$
which is an $\mathfrak{sl}(\alpha)$-submodule by a proposition above.

1. Clear: $\beta(h_\alpha)$ is an eigenvalue of h_α acting on L_β, so it must be in $\mathbb{Z}$ by the proposition above.
2. By a proposition above, $\dim L_{\beta+c\alpha} \in \{0, 1\}$, and the eigenvalues of $\mathrm{ad}(h_\alpha)$ on M are $\beta(h_\alpha) + 2c$. These are all either even or odd, implying M is an irreducible $\mathfrak{sl}(2, \mathbb{C})$-module V_d for some d. So the eigenvalues form the string:
$$\{\beta(h_\alpha) + 2c : \beta + c\alpha \in \Phi\} = \{d, d-2, \ldots, -d\}.$$
This gives part (2), with:
$$d = \beta(h_\alpha) + 2q, \quad -d = -\beta(h_\alpha) - 2r.$$
Adding gives $\beta(h_\alpha) = r - q$.
3. Suppose $\alpha + \beta \in \Phi$ but $[e_\alpha, e_\beta] = 0$. Then $h_\alpha e_\beta = de_\beta$, so $\beta(h_\alpha) = d$, hence $q = 0$. But this contradicts $\alpha + \beta \in \Phi \Rightarrow q \geq 1$, so $[e_\alpha, e_\beta] \neq 0$.
4. This follows from the explicit form of the string in (2), since $\beta - \beta(h_\alpha)\alpha = \beta - (r - q)\alpha \in \Phi$.

□

11.5 Cartan subalgebras as inner product spaces

Proposition 11.32
Let L be semisimple and H a Cartan subalgebra. For $h_1, h_2 \in H$,
$$K(h_1, h_2) = \sum_{\alpha \in \Phi} \alpha(h_1)\alpha(h_2).$$

Proof. Let B be a basis of L, given by:

$$B = (\text{basis of } H) \cup \{e_\alpha : \alpha \in \Phi\}.$$

For $h \in H$, the matrix of $\mathrm{ad}(h)$ in the basis B is:

$$(\mathrm{ad}\, h)_B = \begin{pmatrix} 0 & & & & \\ & \ddots & & & \\ & & 0 & & \\ & & & \alpha(h) & \\ & & & & \ddots \end{pmatrix},$$

where the first block (corresponding to H) contributes 0, and each root space L_α contributes $\alpha(h)I$. So,

$$K(h_1, h_2) = \mathrm{Tr}((\mathrm{ad}\, h_1)(\mathrm{ad}\, h_2)) = \sum_{\alpha \in \Phi} \alpha(h_1)\alpha(h_2),$$

as required. □

Proposition 11.33
We have that $\mathrm{Span}(\Phi) = H^*$, the dual space of H.

Proof. Suppose, for contradiction, that $\mathrm{Span}(\Phi) = W \subsetneq H^*$. Then the annihilator

$$\mathrm{Ann}_H(W) = \{h \in H : f(h) = 0 \text{ for all } f \in W\}$$

is nonzero, since $\dim H^* - \dim W > 0$. Hence, there exists $0 \neq h \in H$ such that $\alpha(h) = 0$ for all $\alpha \in \Phi$. Now,

- $K(h, H) = 0$ by Proposition 12.1, since $\alpha(h) = 0$ for all roots.
- $K(h, L_\alpha) = 0$ for all α, by a previous proposition.

Thus $K(h, L) = 0$, implying $h \in L^\perp = 0$ by non-degeneracy of K. This is a contradiction. Therefore, $\mathrm{Span}(\Phi) = H^*$. □

Proposition 11.34. For each $\alpha \in \Phi$, there exists a unique $t_\alpha \in H$ such that

$$\alpha(x) = K(t_\alpha, x) \quad \text{for all } x \in H.$$

Proposition 11.35. Let $\alpha \in \Phi$ and $x \in L_\alpha, y \in L_{-\alpha}$. Then

$$[xy] = K(x, y)t_\alpha.$$

Proof. For all $h \in H$,

$$K(h, [x, y]) = K([h, x], y) = \alpha(h)K(x, y) = K(h, K(x, y)t_\alpha).$$

Since K is non-degenerate on H, we conclude $[x, y] = K(x, y)t_\alpha$. □

Proposition 11.36.

1. $t_\alpha = \frac{h_\alpha}{K(e_\alpha, e_{-\alpha})}$, and hence $h_\alpha = \frac{2t_\alpha}{K(t_\alpha, t_\alpha)}$.
2. $K(t_\alpha, t_\alpha)K(t_\alpha, h_\alpha) = 4$.

Proposition 11.37. If $\alpha, \beta \in \Phi$, then:

1. $K(h_\alpha, h_\beta) \in \mathbb{Z}$,
2. $K(t_\alpha, t_\beta) \in \mathbb{Q}$.

Proof. We prove each statement in turn.

1. By a previous proposition
$$K(h_\alpha, h_\beta) = \sum_{\gamma \in \Phi} \gamma(h_\alpha)\gamma(h_\beta) \in \mathbb{Z},$$
since all terms are integers.
2. By a previous proposition
$$K(t_\alpha, t_\beta) = K(h_\alpha, h_\beta) \cdot \frac{K(t_\alpha, t_\alpha)}{2} \cdot \frac{K(t_\beta, t_\beta)}{2} \in \mathbb{Q},$$
as required.

□

Proposition 11.38

We define a bilinear form on H^* as follows: for $\theta_1, \theta_2 \in H^*$, let $\theta_i = \theta_{h_i}$ for $h_i \in H$, and define
$$(\theta_1, \theta_2) := K(h_1, h_2).$$
Since K_H is non-degenerate on H, this form is non-degenerate on H^* as well.

For $\alpha, \beta \in \Phi$, we write
$$(\alpha, \beta) := K(t_\alpha, t_\beta).$$

Proposition 11.39. If $\beta \in \Phi$, then
$$\beta = \sum_{i=1}^{k} r_i \alpha_i, \quad \text{for } r_i \in \mathbb{Q}.$$

Proof. Let $\beta = \sum r_i \alpha_i$ for $r_i \in \mathbb{C}$. Then for each j,
$$(\beta, \alpha_j) = \sum_{i=1}^{k} r_i (\alpha_i, \alpha_j).$$
In matrix form:
$$\begin{pmatrix} (\beta, \alpha_1) \\ \vdots \\ (\beta, \alpha_k) \end{pmatrix} = A \begin{pmatrix} r_1 \\ \vdots \\ r_k \end{pmatrix}, \quad \text{where } A = ((\alpha_i, \alpha_j)).$$
By Proposition 12.6, all entries of $A \in \mathbb{Q}$, and since K is non-degenerate, A is invertible. Hence $A^{-1} \in \mathbb{Q}$, so $r_i \in \mathbb{Q}$ for all i. □

Definition 11.40. Let $E \subset H^*$ be the real span of $\alpha_1, \ldots, \alpha_k$, i.e.,

$$E := \left\{ \sum_{i=1}^{k} r_i \alpha_i : r_i \in \mathbb{R} \right\}.$$

Proposition 11.41

Properties.

1. E does not depend on the choice of basis $\alpha_1, \ldots, \alpha_k$,
2. $\Phi \subset E$,
3. $E = \mathrm{Span}_{\mathbb{R}}(\Phi)$.

Proposition 11.42

The bilinear form $(\ , \ )$ is a real-valued inner product on the vector space E.

Proof. For $\alpha, \beta \in \Phi$, we know that

$$(\alpha, \beta) = K(t_\alpha, t_\beta) \in \mathbb{R}.$$

Hence, $(\ , \ )$ is real-valued on E.

To see positive-definiteness, let $\theta \in E$, so $\theta = \theta_h$ for some $h \in H$. Then

$$(\theta_h, \theta_h) = K(h, h) = \sum_{\gamma \in \Phi} \gamma(h)^2$$

This shows that $(\theta_h, \theta_h) \geq 0$ and is zero only when $\gamma(h) = 0$ for all $\gamma \in \Phi$, i.e., $h = 0$, so $\theta = 0$. Therefore, $(\ , \ )$ is a real inner product on E. $\square$

12 Root systems — lecture notes

Let $\theta \in E$, so $\theta = \theta_h$ for some $h \in H$. Then

$$\begin{aligned}(\theta_h, \theta_h) &= K(h,h) \\ &= \sum_{\gamma\in\Phi} \gamma(h)^2 && \text{by Proposition 12.1} \\ &= \sum_{\gamma\in\Phi} K(t_\gamma, h)^2 && \text{by definition of } t_\gamma \\ &= \sum_{\gamma\in\Phi} (\gamma, \theta_h)^2 && \text{by definition of } (\ ,\ )\end{aligned}$$

As $(\gamma, \theta_h) \in \mathbb{R}$, this shows that $(\theta_h, \theta_h) \geq 0$.

If $(\theta_h, \theta_h) = 0$ then $\gamma(h) = 0$ for all $\gamma \in \Phi$. Hence $h = 0$, so $\theta = 0$. □

13. Root systems

Let E be a finite dimensional real vector space with an inner product $(\ ,\ )$. For $0 \neq v \in E$, the *reflection* $s_v \colon E \to E$ is defined by

$$s_v(x) = x - \frac{2(v,x)}{(v,v)} v \text{ for all } x \in E.$$

Note that s_v sends $v \mapsto -v$ and fixes every vector in $v^\perp$.

Note. The reflection s_v preserves the inner product, i.e.

$$(s_v(x), s_v(y)) = (x,y) \text{ for all } x, y \in E.$$

(Showing this is left as an exercise.)

Notation. Write $\langle x, v\rangle = \frac{2(x,v)}{(v,v)}$ (linear in x, not in v).

Definition. A subset R of E is a *root system* if

(1) R is finite, $0 \notin R$, and $\mathrm{Span}(R) = E$,
(2) for $\alpha \in R$, the only scalar multiples of α in R are $\pm\alpha$,
(3) for $\alpha \in R$, the reflection s_α sends R to R, i.e. permutes the set R,
(4) for $\alpha, \beta \in R$,

$$\langle \beta, \alpha\rangle = \frac{2(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}.$$

We call the elements of R *roots*, and $\dim E$ the rank of R.

Example. The only root system of rank 1 is

Proposition 13.1. *Let L be a semisimple Lie algebra over $\mathbb{C}$, H a Cartan subalgebra, and root decomposition*

$$L = H \oplus \bigoplus_{\alpha\in\Phi} L_\alpha.$$

Let $E = \mathrm{Span}_{\mathbb{R}}(\Phi) \subseteq H^*$, with inner product $(\ ,\ )$ as in Chapter 12. Then Φ is a root system in E.

Proof. Axiom (1) is clear. Axiom (2) is Proposition 11.3 (2), and Sheet 5, Question 2.

To check axiom (3), let $\alpha, \beta \in \Phi$. Then

$$s_\alpha(\beta) = \beta - \langle \beta, \alpha \rangle \alpha.$$

We claim that $\langle \beta, \alpha \rangle = \beta(h_\alpha)$. Indeed:

$$\begin{aligned}
\beta(h_\alpha) &= K(t_\beta, h_\alpha) && \text{by definition}\\
&= K\left(t_\beta, \tfrac{2t_\alpha}{K(t_\alpha,t_\alpha)}\right) && \text{by Proposition 12.5}\\
&= \tfrac{2(\beta,\alpha)}{(\alpha,\alpha)} && \text{by definition}\\
&= \langle \beta, \alpha \rangle && \text{by definition}
\end{aligned}$$

Hence $s_\alpha(\beta) = \beta - \beta(h_\alpha)\alpha \in \Phi$ by Proposition 11.5 (4). Finally, (4) follows from $\langle \beta, \alpha \rangle = \beta(h_\alpha) \in \mathbb{Z}$ by Proposition 11.5 (1). □

Examples.

(1) Let $L = \mathfrak{sl}(n, \mathbb{C})$. The Cartan subalgebra H consists of diagonal matrices in L, the root spaces are

$$\mathrm{Span}(E_{ij}) \text{ with roots } \epsilon_i - \epsilon_j \quad (i \neq j),$$

where

$$\epsilon_i \colon \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} \mapsto a_i.$$

Here

$$E = \mathrm{Span}_{\mathbb{R}}(\Phi) = \left\{ \sum_{i=1}^n \lambda_i \epsilon_i : \sum_{i=1}^n \lambda_i = 0 \right\}$$

of dimension $n-1$. Inner product (rescaling) is the usual

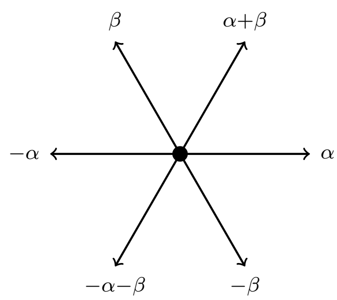
$$\left(\sum \lambda_i \epsilon_i, \sum \mu_i \epsilon_i \right) = \sum \lambda_i \mu_i.$$

(2) **Some rank 2 root systems.**

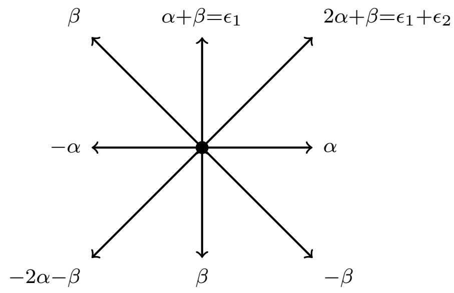
(a) Let $L = \mathfrak{sl}(3, \mathbb{C})$. Let $\alpha = \epsilon_1 - \epsilon_2$, $\beta = \epsilon_2 - \epsilon_3$. Then the root system

$$\Phi = \{\alpha, \beta, \alpha+\beta, -\alpha, -\beta, -\alpha-\beta\}.$$

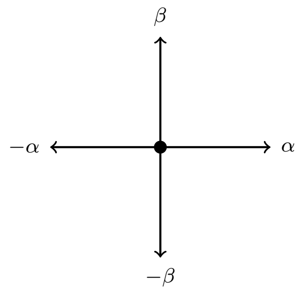
Also $(\alpha, \beta) = -1$, $(\alpha, \alpha) = 2$. So the angle between α and β is $\cos^{-1}\left(\frac{1}{2}\right) = \frac{2\pi}{3}$.



(b) Another rank 2 root system. Again, let ϵ_1, ϵ_2 be standard unit vectors in $\mathbb{R}^2$, and let $\alpha = \epsilon_2$, $\beta = \epsilon_1 - \epsilon_2$. The angle between α and β is $\cos^{-1}\left(\frac{-1}{\sqrt{2}}\right) = \frac{3\pi}{4}$.



(c) Another one



Definition. We say that root systems $R \subseteq E$, $R' \subseteq E'$ are *isomorphic* if there exists a vector space isomorphism $\phi\colon E \to E'$ such that

(1) $\phi(R) = R'$,
(2) $(\phi(\alpha), \phi(\beta)) = (\alpha, \beta)$ for any $\alpha, \beta \in R$.

Definition. A root system $R \subseteq E$ is *reducible* if $R = R_1 \cup R_2$ where $R_i \neq \emptyset$ and

$$(\alpha, \beta) = 0 \text{ for any } \alpha \in R_1, \beta \in R_2.$$

Otherwise, R is *irreducible.*

Example. In the examples above, (2)(c) is reducible, but (2)(a) and (2)(b) are irreducible.

Proposition 13.2. *Let L be a semisimple Lie algebra over $\mathbb{C}$ with root system of Φ. If Φ is irreducible, then L is simple.*

Proof. We have

$$L = H \oplus \bigoplus_{\alpha\in\Phi} L_\alpha.$$

Suppose L is not simple, so it has an ideal $I \neq 0, L$. Now, $[HI] \subseteq I$, and $\operatorname{ad} H$ is simultaneously diagonalizable on L, hence also on I.

Therefore, I has a basis of common eigenvectors for $\operatorname{ad} H$, so

$$I = H_1 \oplus \bigoplus_{\alpha\in\Phi_1} L_\alpha$$

where $H_1 \subseteq H$, $\Phi_1 \subseteq \Phi$. Similarly,

$$I^\perp = H_2 \oplus \bigoplus_{\alpha \in \Phi_2} L_\alpha.$$

As $L = I \oplus I^\perp$,

$$H = H_1 \oplus H_2,\ \Phi = \Phi_1 \cup \Phi_2,\ \Phi_1 \cap \Phi_2 = \emptyset.$$

If $\Phi_2 = \emptyset$, then $\Phi_1 = \Phi$, so I contains all L_α ($\alpha \in \Phi$), so also all $[L_\alpha L_{-\alpha}]$, which span H. Hence $I = L$, a contradiction. Hence $\Phi_i \neq \emptyset$ for $i = 1, 2$.

Finally, for $\alpha \in \Phi_1$, $\beta \in \Phi_2$,

$$[h_\beta e_\alpha] \in I \cap I^\perp = 0,$$

so

$$0 = \alpha(h_\beta) = \langle \alpha, \beta \rangle$$

by the proof of Proposition 13.1. Hence $(\alpha, \beta) = 0$, showing that Φ is reducible. □

Classification theorems (Killing, Cartan). The root system Φ depends on the choice of Cartan subalgebra H. However, we have the following theorem.

Theorem 13.3. *Let L be a semisimple Lie algebra over $\mathbb{C}$ with Cartan subalgebras H_1, H_2 and corresponding root systems Φ_1, Φ_2. Then $\Phi_1 \cong \Phi_2$.*

The proof is based on the fact that all Cartan subalgebras are *conjugate*, i.e. there exists

$$g \in \mathrm{Aut}(L) = \{x \in \mathrm{GL}(L) : x([ab]) = [x(a), x(b)] \text{ for all } a, b \in L\}$$

such that $g(H_1) = H_2$.

So every semisimple Lie algebra over $\mathbb{C}$ has a unique unique root system. The converse also holds.

Theorem 13.4. *For any root system Φ, there exists a unique (up to isomorphism) semisimple Lie algebra L over $\mathbb{C}$ with root system Φ.*

Uniqueness. We can specify the structure constants of L in terms of root systems Φ: there exists a *Chevalley* basis of L with the following structure constants. Recall, for $\alpha, \beta \in \Phi$, the α-string through β is

$$\beta - r\alpha, \ldots, \beta + q\alpha.$$

Then the Chevalley basis is h_α's, e_α's, and the structure constants are

$$[h_\alpha h_\beta] = 0$$

$$[h_\alpha e_\beta] = \beta(h_\alpha) e_\beta \text{ and } \beta(h_\alpha) = r - q$$

$$[e_\alpha e_{-\alpha}] = h_\alpha$$

$$[e_\alpha e_\beta] = \begin{cases} 0 & \text{if } \alpha + \beta \notin \Phi \cup \{0\} \\ \pm(q+1) e_{\alpha+\beta} & \text{if } \alpha + \beta \in \Phi \end{cases}$$

Existence. Given an irreducible root system (these are classified), one can construct a simple Lie algebra with that root system. The simple Lie algebras as

$$\begin{array}{rl} \text{(classical)} & \mathfrak{sl},\ \mathfrak{sp},\ \mathfrak{so}, \\ \text{(exceptional)} & \mathfrak{g}_2,\ \mathfrak{f}_2,\ \mathfrak{e}_6,\ \mathfrak{e}_7,\ \mathfrak{e}_8. \end{array}$$

Alternative approach to existence: use *Serre relations*.

Conclusion. Classification of simple Lie algebras is equivalent to classification of irreducible root systems.

14. Irreducible root systems

Let $R \subseteq E$ be a root system for a real inner product space E. Recall that for $\alpha, \beta \in R$,

$$\langle \alpha, \beta \rangle = \frac{2(\beta, \alpha)}{(\alpha, \alpha)}.$$

Also,

$$(\alpha, \alpha) = ||\alpha||^2$$

(where $||\alpha||$ is the length of α), and

$$(\alpha, \beta) = ||\alpha|| \, ||\beta|| \cos \theta$$

where θ is the angle between α and β.

Proposition 14.1. *If $\alpha, \beta \in R$ and $\beta \neq \pm\alpha$, then*

$$\langle \beta, \alpha \rangle \langle \alpha, \beta \rangle = \frac{4(\alpha, \beta)^2}{(\alpha, \alpha)(\beta, \beta)} \in \{0, 1, 2, 3\}.$$

Proof. If θ is the angle between α and β,

$$\langle \beta, \alpha \rangle \langle \alpha, \beta \rangle = 4 \cos^2 \theta \leq 4.$$

We only have to note that it is not equal to 4; otherwise, $\cos^2 \theta = 1$, and $\theta = n\pi$, so $\beta = \pm\alpha$. □

Proposition 14.2. *Let $\alpha, \beta \in R$, $\beta \neq \pm\alpha$, and assume that $(\beta, \beta) \geq (\alpha, \alpha)$. The possibilities for $\langle \alpha, \beta \rangle$, $\langle \beta, \alpha \rangle$, θ are as follows*

$\langle \alpha, \beta \rangle$	$\langle \beta, \alpha \rangle$	$\cos \theta$	θ	$\frac{(\beta,\beta)}{(\alpha,\alpha)} = \frac{\langle \beta,\alpha \rangle}{\langle \alpha,\beta \rangle}$
0	0	0	$\frac{\pi}{2}$	
1	1	$\frac{1}{2}$	$\frac{\pi}{2}$	1
-1	-1	$-\frac{1}{2}$	$\frac{2\pi}{3}$	1
1	2	$\frac{1}{\sqrt{2}}$	$\frac{\pi}{4}$	2
-1	-2	$-\frac{1}{\sqrt{2}}$	$\frac{3\pi}{4}$	2
1	3	$\frac{\sqrt{3}}{2}$	$\frac{\pi}{6}$	3
-1	-3	$-\frac{\sqrt{3}}{2}$	$\frac{5\pi}{6}$	3

The proof is an easy calculation.

Proposition 14.3. *Let θ be the angle between $\alpha, \beta \in R$.*

(1) *If $\theta > \frac{\pi}{2}$, then $\alpha + \beta \in R$.*
(2) *If $\theta < \frac{\pi}{2}$, then $\alpha - \beta \in R$.*

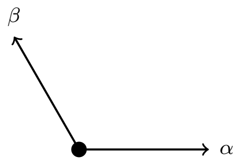
Proof. By axiom (3),

$$s_\beta = \alpha - \langle \alpha, \beta \rangle \beta \in R.$$

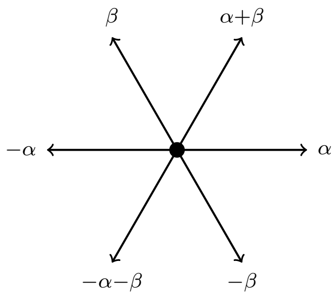
From Proposition 14.2, $\theta > \frac{\pi}{2}$ implies that $\langle \alpha, \beta \rangle = -1$, and $\theta < \frac{\pi}{2}$ implies that $\langle \alpha, \beta \rangle = 1$. □

Example (Classification of root systems of rank 2)**.** Let $R \subseteq \mathbb{R}^2$ be a rank 2 root system. Pick $\alpha, \beta \in R$ with $\beta \neq \pm\alpha$, and the angle θ between α and β as large as possible ($\theta \geq \frac{\pi}{2}$). By Proposition 14.2, the possibilities of θ are $\frac{2\pi}{3}$, $\frac{3\pi}{4}$, $\frac{5\pi}{6}$ or $\frac{\pi}{2}$.

Take $\theta = \frac{2\pi}{3}$. Then α, β have the same length, and we have

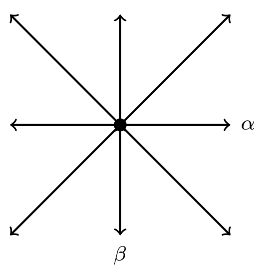


Apply reflections to get the root system



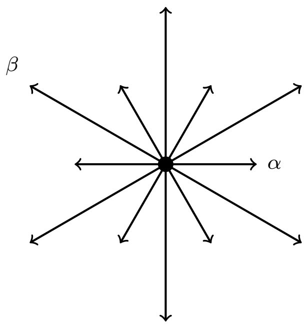
A_2

Take $\theta = \frac{3\pi}{4}$. Here, $(\beta, \beta) = 2(\alpha, \alpha)$, and we get



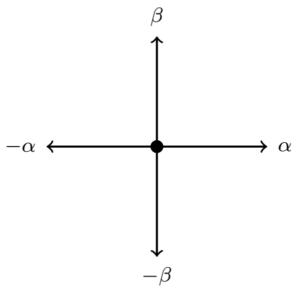
B_2

Take $\theta = \frac{5\pi}{6}$. Here, $(\beta, \beta) = 3(\alpha, \alpha)$, and we get



G_2

Take $\theta = \frac{\pi}{2}$. Here, we get



$A_1 + A_1$

This root system is *reducible*.

Weyl groups. Recall that for a root system $R \subseteq E$ and a root $\alpha \in R$, we defined the reflection s_α by

$$s_\alpha(x) = x - \frac{2(\alpha, x)}{(\alpha, \alpha)}\alpha \text{ for all } x \in E.$$

Definition. The *Weyl group* $W(R)$ of a root system $R \subseteq E$ is

$$W(R) = \langle s_\alpha : \alpha \in R \rangle$$

a subgroup of $\mathrm{GL}(E)$.

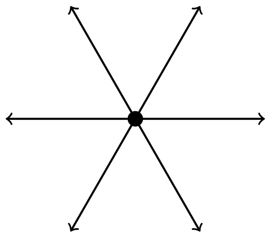
Proposition 14.4. *The Weyl group $W(R)$ is finite.*

Proof. By axioms (3), each reflection s_α gives a permutation of the finite set R. So we have a homomorphism

$$\phi\colon W(R) \to \mathrm{Sym}(R).$$

If $g \in \operatorname{Ker}\phi$, then $g(\alpha) = \alpha$ for all $\alpha \in R$, hence as $E = \mathrm{Span}(R)$, $g = 1$. Therefore, $\operatorname{Ker}\phi = 1$, so $W(R) \cong \operatorname{Im}\phi \leq \mathrm{Sym}(R)$. □

Examples. $\underline{R = A_2}$



Here $W(A_2) = \langle s_\alpha, s_\beta, s_{\alpha+\beta}\rangle$. Action on basis $\epsilon_1, \epsilon_2, \epsilon_3$ of $\mathbb{R}^3$:

$$s_\alpha = s_{\epsilon_1-\epsilon_2} : \ \epsilon_1 \leftrightarrow \epsilon_2, \ \epsilon_3 \mapsto \epsilon_3$$

$$s_\beta : \epsilon_2 \leftrightarrow \epsilon_3, \ \epsilon_1 \mapsto \epsilon_1$$

$$s_{\alpha+\beta} : \epsilon_1 \leftrightarrow \epsilon_3, \epsilon_2 \mapsto \epsilon_2$$

Hence $W(A_2) \cong S_3$.

$\underline{R = A_{n-1}}$ Roots $\epsilon_i - \epsilon_j$ $(i \neq j)$ for $i, j \in \{1, \dots, n\}$. Reflection $s_{\epsilon_i - \epsilon_j}$ sends $\epsilon_i \leftrightarrow \epsilon_j$, and fixes the other basis vectors. Hence $W(A_{n-1}) \cong S_n$.

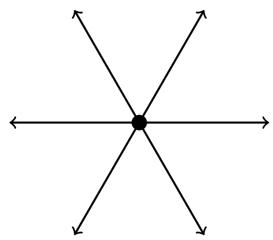
Bases.

Definition. Let $R \subseteq E$ be a root system. We say a subset B of R is a *base* of R if

(1) B is a basis of E (as a vector space)
(2) for any $\beta \in R$,
$$\beta = \sum_{\alpha \in B} n_\alpha \alpha$$
where $n_\alpha \in \mathbb{Z}$, and either $n_\alpha \geq 0$ for all α or $n_\alpha \leq 0$ for all α.

We say $\beta \in R$ is a *positive root* (with respect to base B) if all $n_\alpha \geq 0$. Similarly, define *negative roots*.

Example. $\underline{R = A_2}$



A base is α, β (the positive roots are α, β, $\alpha + \beta$).

Another base is $\alpha, -\alpha - \beta$ (the positive roots are $\alpha, -\alpha - \beta, -\beta$).

Note, there exists $w \in W(A_2)$ sending $\{\alpha, \beta\}$ to $\{\alpha, -\alpha - \beta\}$. (Exercise)

Theorem 14.5.

(1) *Every root system has a base.*
(2) *If B, B' are bases of R, then there exists a unique $w \in W(R)$ such that $w(B) = B'$.*

Hence R has precisely $|W(R)|$ different basis.

Example. The root system A_2 has 6 different bases, all of the form $w(\{\alpha, \beta\})$ for $w \in W(A_2)$.

Proof of Theorem 14.5. Omitted. □

Dynkin diagrams. Let R be a root system with a base B. Define the *Dynkin diagram* $\Delta = \Delta(R)$ of R (with respect to B) to be the following graph[5]

vertices: elements of B
edges: join α, β in B by $d_{\alpha\beta}$ edges, where $d_{\alpha\beta} = \langle \beta, \alpha \rangle \in \{0, 1, 2, 3\}$

If $d_{\alpha\beta} > 1$, then α, β have different lengths, and we draw an arrow from the longer to the shorter root.

Note. By Theorem 14.5, the diagram Δ does not depend on the choice of the base B.

Examples.

(1) A_{n-1}. Roots $\epsilon_i - \epsilon_j$ for $i \neq j$ in $\{1, \dots, n\}$. Here is a base:

$$\underbrace{\epsilon_1 - \epsilon_2}_{\alpha_1}, \ \underbrace{\epsilon_2 - \epsilon_3}_{\alpha_2}, \ \dots, \ \underbrace{\epsilon_{n-1} - \epsilon_n}_{\alpha_{n-1}}$$

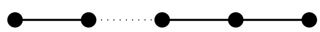
This **is** a base, as for $i < j$,

$$\epsilon_i - \epsilon_j = \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1}.$$

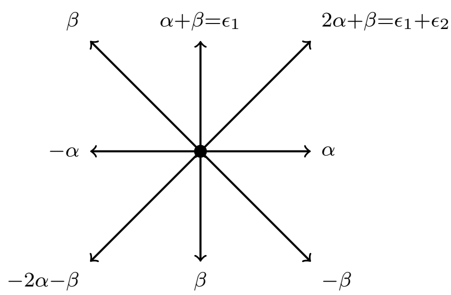
So the positive roots are $\epsilon_i - \epsilon_j$ for $i < j$ and the negative roots are $\epsilon_i - \epsilon_j$ for $i > j$. Here,

$$d_{\alpha_i \alpha_{i+1}} = 1$$
$$d_{\alpha_i \alpha_j} = 0 \text{ if } j \neq i \pm 1.$$

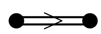
So the Dynkin diagram is



(2) B_2.

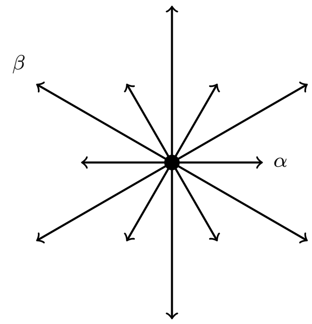


It has base α, β, and its Dynkin diagram is

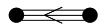


[5]A graph where we allow multiple edges between vertices.

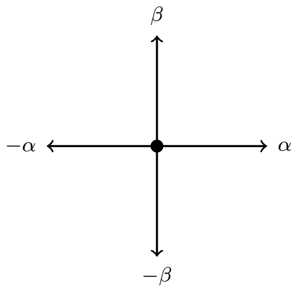
(3) G_2.



It has base α, β, and its Dynkin diagram is



(4) $A_1 + A_1$.



It has base α, β, and its Dynkin diagram is

Proposition 14.6. *A root system R is irreducible if and only if its Dynkin diagram is connected.*

Proof. Sheet 5, Question 3. □

Simplicity of classical Lie algebras.

Theorem 14.7. *Let $n \geq 2$. Then the classical Lie algebras, $\mathfrak{sl}(n, \mathbb{C})$, $\mathfrak{so}(n, \mathbb{C})$, $\mathfrak{sp}(n, \mathbb{C})$ (n even) are simple, apart from $\mathfrak{so}(2, \mathbb{C})$ and $\mathfrak{so}(4, \mathbb{C})$.*

Idea.

(1) Use the next proposition to show L is semisimple.
(2) Use the root system and Proposition 14.6 to show L is simple.

Proposition 14.8. *Let L be a finite-dimensional Lie algebra over $\mathbb{C}$, with $Z(L) = 0$. Assume H is a Cartan subalgebra and*

$$L = H \oplus \bigoplus_{\alpha \in \Phi} L_\alpha$$

where $\Phi \subseteq H^* \setminus \{0\}$. Suppose

(1) $\dim L_\alpha = 1$ *for all* $\alpha \in \Phi$

(2) *if* $\alpha \in \Phi$ *then* $-\alpha \in \Phi$
(3) $[[L_\alpha, L_{-\alpha}], L_\alpha] \neq 0$ *for all* $\alpha \in \Phi$.

Then L *is semisimple.*

Proof. Suppose for a contradiction that L is not semisimple. Then L has a soluble ideal, so it has an abelian ideal $I \neq 0$. Now $[HI] \subseteq I$, and $\operatorname{ad} H$ acts diagonalizably on L, hence also on I.

Hence I is a sum of eigenspaces of $\operatorname{ad} H$, so

$$I = (I \cap H) \oplus \sum_{\alpha \in \Phi} (I \cap L_\alpha).$$

If $I \cap L_\alpha \neq 0$ for some α, then $L_\alpha \subseteq I$ by (1), so

$$[[L_\alpha, L_{-\alpha}], L_\alpha] \subseteq [[I, L_{-\alpha}], I] \subseteq [I, I] = 0,$$

a contradiction with (3).

Therefore, $I = I \cap H$, i.e. $I \subseteq H$. As $Z(L) = 0$, there exists $\alpha \in \Phi$ such that $[IL_\alpha] \neq 0$. But

$$[IL_\alpha] \subseteq I \cap [HL_\alpha] \subseteq I \cap L_\alpha = 0,$$

a contradiction. □

Proof of Theorem 14.7. We just do the case $L = \mathfrak{sl}(n, \mathbb{C})$. The other cases $\mathfrak{so}$, $\mathfrak{sp}$ are Sheet 5, Question 7.

We use Proposition 14.8. First, the case $Z(L) = 0$ was Sheet 3, Question 3. Otherwise, we have a Cartan subalgebra H of diagonal matrices, and root space decomposition

$$L = H \oplus \bigoplus_{\alpha \in \Phi} L_\alpha.$$

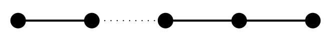
We check assumptions (1)–(3) of Proposition 14.8:

(1) the root spaces L_α are of the form $\operatorname{Span}(E_{ij})$, 1-dimensional,
(2) $\epsilon_i - \epsilon_j \in \Phi$ implies that $\epsilon_j - \epsilon_i \in \Phi$
(3) for $L_\alpha = \operatorname{Span}(E_{ij})$, we have $L_{-\alpha} = \operatorname{Span}(E_{ji})$, and check

$$[[E_{ij}, E_{ji}], E_{ij}] = 2E_{ij} \neq 0.$$

Hence L is semisimple by Proposition 14.8.

Finally, the Dynkin diagram of L is



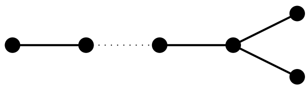
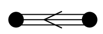
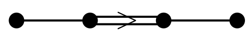
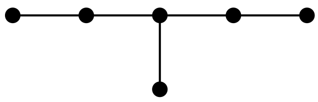
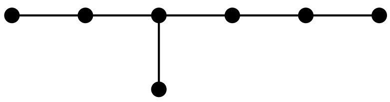
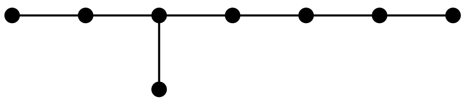
which is connected, so L is simple by Propositions 13.2 and 14.6. □

Classification of irreducible root systems.

Theorem 14.9. *Let $R \subseteq E$ and $R' \subseteq E'$ be roots, and suppose R, R' have the same Dynkin diagram. Then $R \cong R'$.*

So we just need to classify the possible Dynkin diagrams.

Theorem 14.10. *The Dynkin diagrams of the irreducible root systems are*

Dynkin diagram	*Notation*	*Corresponding simple Lie algebra*
	A_n $(n \geq 1)$	$\mathfrak{sl}(n+1, \mathbb{C})$
	B_n $(n \geq 2)$	$\mathfrak{so}(2n+1, \mathbb{C})$
	C_n $(n \geq 2)$	$\mathfrak{sp}(2n, \mathbb{C})$
	D_n $(n \geq 3)$	$\mathfrak{so}(2n, \mathbb{C})$
	G_2	$\mathfrak{g}_2$
	F_4	$\mathfrak{f}_2$
	E_6	$\mathfrak{e}_6$
	E_7	$\mathfrak{e}_7$
	E_8	$\mathfrak{e}_8$

In particular, each of these Dynkin diagram shows the existence of the Lie algebra on the right. However, these Lie algebras are actually not easy to construct.

We will prove Theorem 14.9. For that sake, we need the following proposition.

Proposition 14.11. *Let $R \subseteq E$ be a root system, with base B. Define*

$$W_0 = \langle s_\alpha : \alpha \in B \rangle \leq W(R).$$

If $\beta \in R$, then there exists $\alpha \in B$, $w \in W_0$ such that $w(\alpha) = \beta$ (i.e. $W_0(B) = R$).

Proof. Suppose $\beta \in R^+$ (positive root with respect to B), so

$$\beta = \sum_{\gamma \in B} k_\gamma \gamma, \; k_\gamma \geq 0.$$

Define

$$\mathrm{ht}(\beta) = \sum_{\gamma \in B} k_\gamma.$$

We proceed by induction on $\mathrm{ht}(\beta)$. If $\mathrm{ht}(\beta) = 1$, then $\beta \in B$ and take $\alpha = \beta$, $w = 1$.

Now, assume $\mathrm{ht}(\beta) \geq 2$, By axiom (2) of a root system, at least two k_γ's are nonzero.

We first claim that there exists $\gamma_0 \in B$ such that $(\beta, \gamma_0) > 0$. Otherwise,

$$(\beta, \beta) = \sum_{\gamma \in B} k_\gamma(\beta, \gamma) \leq 0,$$

so $(\beta, \beta) = 0$, hence $\beta = 0$, contradicting $\beta \in R$.

We now claim that $s_{\gamma_0}(\beta) \in R^+$. Well,

$$s_{\gamma_0}(\beta) = \beta - \langle \beta, \gamma_0 \rangle \gamma_0,$$

so $s_{\gamma_0}(\beta)$ has at least one coefficient $k_\gamma > 0$, and hence $s_{\gamma_0}(\beta) \in R^+$.

By the previous two claims,

$$\mathrm{ht}(s_{\gamma_0}(\beta)) = \mathrm{ht}(\beta) - \langle \beta, \gamma_0 \rangle < \mathrm{ht}(\beta).$$

By the inductive hypothesis, there exists $\alpha \in B$, $w \in W_0$ such that

$$w(\alpha) = s_{\gamma_0}(\beta).$$

Then $s_{\gamma_0} w \in W_0$ and it sends α to β. □

Proof of Theorem 14.9. We have $R \subseteq E$, $R' \subseteq E'$ with bases

$$B = \{\alpha_1, \ldots, \alpha_n\},\ B' = \{\alpha_1', \ldots, \alpha_n'\}$$

such that

$$\langle \alpha_i, \alpha_j \rangle = \langle \alpha_i', \alpha_j' \rangle \text{ for all } i, j$$

(same Dynkin diagram).

Define a linear map $\phi \colon E \to E'$ by

$$\phi(\alpha_i) = \alpha_i' \text{ for all } i.$$

We need to show that $\phi(R) = R'$. (This will show that $R \cong R'$.)

We use Proposition 14.11 to obtain

$$\{w_0(\alpha) : \alpha \in B, w_0 \in W_0\} = R.$$

Now

$$\phi(s_{\alpha_i}(\alpha_j)) = \phi(\alpha_j - \langle \alpha_j, \alpha_i \rangle \alpha_i) = \alpha_j' - \langle \alpha_j', \alpha_i' \rangle \alpha_i' = s_{\alpha_i'}(\alpha_j') \in R'$$

by axiom (3). Hence for $w_0 \in W_0$,

$$\phi(w_0(\alpha)) \in R'.$$

So $\phi(R) \subseteq R'$.

The same argument for ϕ^{-1} gives $\phi^{-1}(R') \subseteq R$. Hence $\phi(R) = R'$. □