# Calculus I Revision Notes

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## Contents

# 1 Standard Notation

$$\begin{aligned}
\mathbb{N} &= \{1, 2, 3 \ldots\} && \text{the set of natural numbers,} \\
\mathbb{Z} &= \{\ldots, -2, -1, 0, 1, 2, \ldots\} && \text{the set of integers,} \\
\mathbb{Q} &= \{\frac{m}{n} : m \in \mathbb{Z}, n \in \mathbb{N}\} && \text{the set of rational numbers,} \\
\mathbb{R} &= (-\infty, \infty) && \text{the set of real numbers,} \\
\mathbb{R}^+ &= (0, \infty) && \text{the set of positive numbers,} \\
\mathbb{R}_0^+ &= [0, \infty) && \text{the set of non-negative real numbers.}
\end{aligned}$$

# 2 Functions of one variable

## 2.1 Definition of a function

**Definition 2.1.** A **function** $f : A \to B$ is a map from a set $A$ to a set $B$ that associates a unique element in $B$ to each element in $A$.

**Notation:** The symbol $\to$ indicates a map between sets, $f : A \to B$, and the symbol $\mapsto$ indicated a map acting on a single element in a set, $x \mapsto y, (x \in A, y \in B)$.

**Remark.** *A function $f : A \to B$ acts on **all** elements in $A$ to give an element in $B$ but it is **not necessary** that every element in $B$ is associated with an element in $A$ by a function.*

## 2.2 Well-defined functions

**Definition 2.2.** A function is said to be **well-defined** if and only if the function is **not** a **many-to-one** function.

The vertical line test can be used to deduce whether a 'function' is well-defined:

**Definition 2.3.** Given a function $f$, every vertical line that may be drawn intersect the graph of $f$ no more than once. If any vertical line intersects a set of points more than once, the set of points does not represent a function.

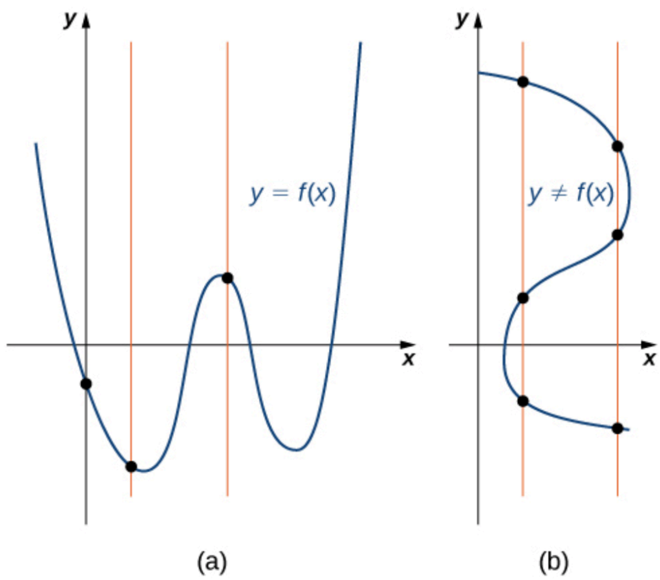


Figure 1: Illustration of vertical line test.

## 2.3 Domain and Range

**Definition 2.4.** For a function $f : A \to B$, the set $A$ is called the **domain** of $f$ and the set $B$ is called the **range** of $f$.

**Terminology:** The set of all elements $f(x)$ for all $x \in A$ is called the **image** of $f$.

## 2.4 Injections, Surjections and Bijections

**Definition 2.5.** A function $f : A \to B$ is **injective** (or **one-to-one**) if every element $f(x) \in B$ is mapped to by at most one element in the domain $A$. An injective function is called an **injection**.

More formally expressed as:

$$f : A \to B,\ \forall x_1, x_2 \in A, f(x_1) = f(x_2) \Rightarrow x_1 = x_2.$$

**Example.** Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by $f(x) = 3x + 7$. Prove that $f$ is injective.
**Solution:** Suppose $f(x) = f(y)$ for some $x, y \in \mathbb{Z}$. So, substituting into $f, 3x + 7 + 3y + 7$. Therefore, $3x = 3y \Rightarrow x = y$. Hence, $f$ is injective.

**Theorem 2.1.** A function $f$ is injective if and only if every horizontal line intersects the graph of $f$ no more than once.

**Remark.** *Variations of the horizontal line test can be used as such:*

- *The function $f$ is surjective if and only if its graph intersects any horizontal line at **least** once.*
- *$f$ is bijective if and only if any horizontal line will intersect the graph **exactly** once.*

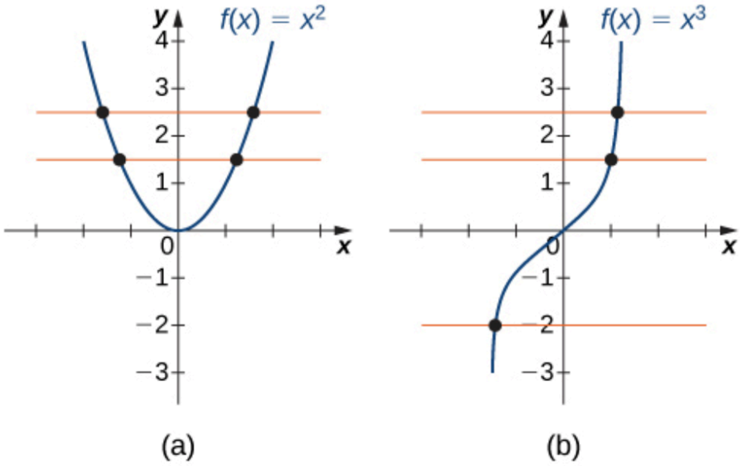


Figure 2: Illustration of the horizontal line test

**Definition 2.6.** A function $f : A \to B$ is **surjective** (or **onto**) if every element in $B$ is mapped to by at least one element of $A$. A surjective function is called a **surjection**.

In mathematical notation the definition is as follows:

$$f : A \to B, \forall y \in B, \exists x \in A \text{ such that } y = f(x).$$

**Example.** Define the function g from the integers to the integers by the formula $g(x) = x - 8$. Prove that $g$ is surjective.
**Solution:** Need to show that for every integer $y$, there is an integer $x$ such that $f(x) = y$. so, choose $x = y + 8$. The chosen value of $x$ is an integer since it is the sum of two integers. Therefore, $g(x) = g(y + 8) = (y + 8) - 8 = y$. The function $g$ is surjective.

**Definition 2.7.** A function $f : A \to B$ is **bijective** if it is both surjective and injective. A bijective function is called a **bijection**.

## 2.5 Inverse functions

**Definition 2.8.** The **inverse function**, denoted $f^{-1}$, of a bijective function $f : A \to B$ is a function defined by

$$f^{-1} : B \to A, \quad f^{-1}(f(a)) = a \quad \forall a \in A.$$

**Remark.** *For a function to have an inverse it must be bijective.*

**Remark.** *The graph of the inverse function is a reflection of the function on the line $y = x$, i.e. the graph is given by the original function but with the $x$-axis and $y$-axis interchanged.*

## 2.6 Composition of functions

**Definition 2.9.** The **composition** $f \circ g$ of functions $g : A \to B$ and $f : B \to C$ is defined as

$$(f \circ g)(x) := f(g(x)), \quad \forall x \in A.$$

**Remark.** *Most of the times* $f \circ g \neq g \circ f$

### 2.6.1 Equal functions

**Definition 2.10.** The functions $f$ and $g$ are said to be **equal** if and only if

- $f$ and $g$ have the same domain;
- $f(x) = g(x)$ for every $x$ in that domain.

**Example.** Is $f(x) = \frac{x^2}{x}$ the same function as $g(x) = x$?
**Solution:** If the domain is $\mathbb{R}$ then no, because $f(0) \neq g(0)$. The domain can be changed so that $f = g$.

## 2.7 Standard functions

### 2.7.1 Even and Odd functions

**Definition 2.11. Even functions** are symmetric about the $y$-axis :

$$f(x) = f(-x) \text{ for all } x.$$

**Definition 2.12. Odd functions** are symmetric about the origin:

$$-f(x) = f(-x) \text{ for all } x.$$

**Remark.** *Common even functions: cosine, hyperbolic cosine and absolute value. Common odd functions: sine, hyperbolic sine, identity function.*

### 2.7.2 Logarithm

**Definition 2.13.** The **logarithmic function**, or logarithm, to the base $a \in \mathbb{R}^+ \backslash \{1\}$, written as $\log(x)$, is the inverse of the function $f : \mathbb{R} \to \mathbb{R}^+$, $f(x) = a^x$.

**Theorem 2.2. Properties of the Logarithm:**

(i) Change of basis, $\log_a(x) = \frac{\log_b(x)}{\log_b(a)}$;

(ii) Additive property, $\log_a(xy) = \log_a(x) + \log_a(y)$;

(iii) Subtractive property, $\log_a(\frac{x}{y}) = \log_a(x) - \log_a(y)$;

(iv) $\log_a(x^y) = y \log_a(x)$.

## 2.8 Trigonometric functions

### 2.8.1 Radians

**Definition 2.14.** One **radian** is the angle subtended at the centre of a circle by an arc that is equal in length to the radius of the circle.

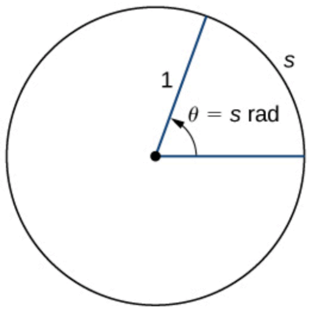


Figure 3: The radian measure of an angle $\theta$.

### 2.8.2 Definition of trigonometric functions

**Definition 2.15.** The functions **cosine** and **sine** are defined by the point $(x, y)$ is at an angle $\theta$ to the $x$-axis are denoted as follows: $(x, y) = (\cos(\theta), \sin(\theta))$.

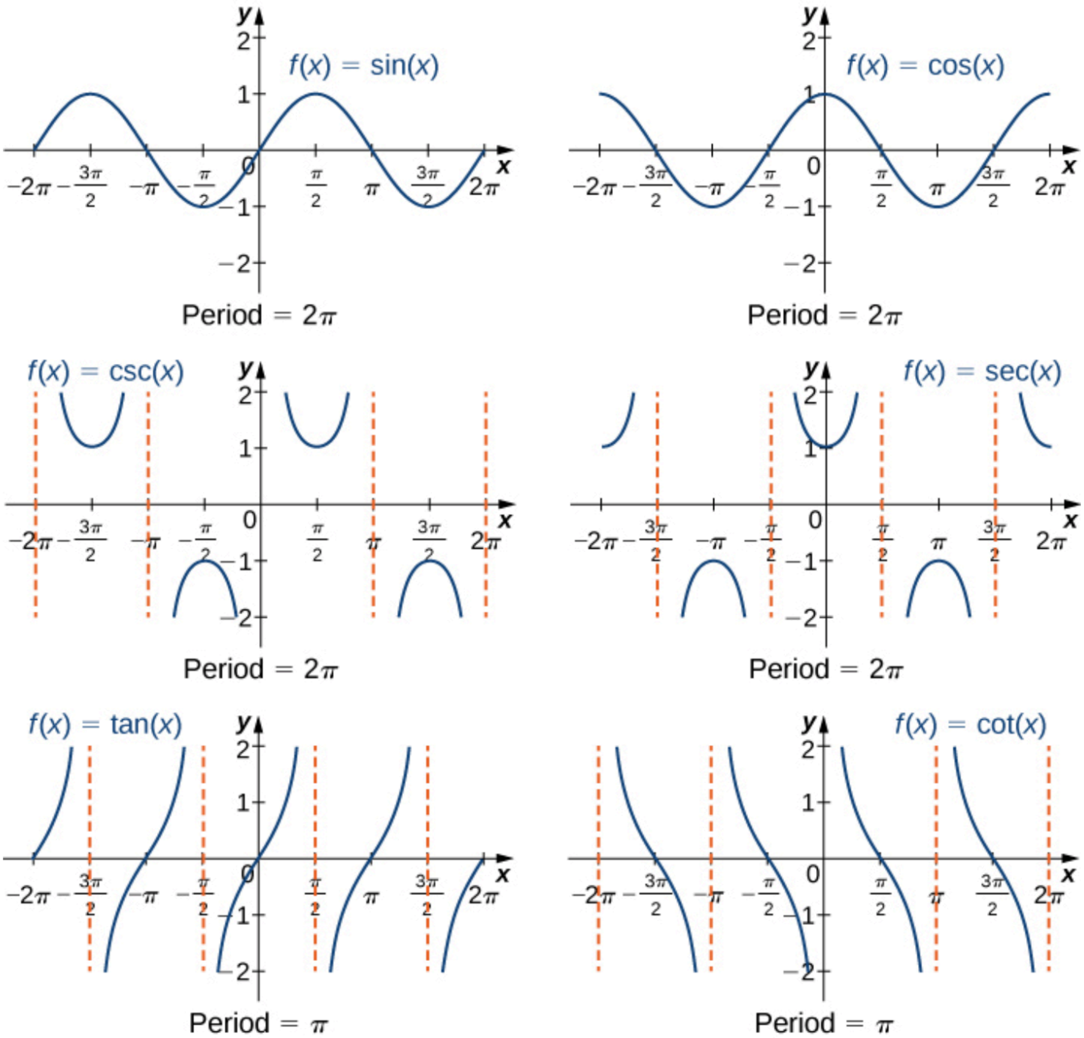


Figure 4: Graphs of trigonometric functions

**Remark.** *The graph of the functions* $\sec(x)$ *and* $\csc(x)$ *sit on the peaks of* $\cos(x)$ *and* $\sin(x)$ *respectively.*

### 2.8.3 Trigonometric Identities

### Even and Odd

1. Cosine is an even function:

$$\cos(\theta) = \cos(-\theta);$$

2. Sine is an odd function:

$$\sin(-\theta) = -\sin(-\theta).$$

**Reciprocal identities**

1.
$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)};$$

2.
$$\cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)};$$

3.
$$\csc(\theta) = \frac{1}{\sin(\theta)};$$

4.
$$\sec(\theta) = \frac{1}{\cos(\theta)}.$$

**Pythagorean identities**

1.
$$\sin^2(\theta) + \cos^2(\theta) = 1;$$

2.
$$1 + \tan^2(\theta) = \sec^2(\theta);$$

3.
$$1 + \cot^2(\theta) = \csc^2(\theta).$$

**Addition and subtraction**

1.
$$\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta);$$

2.
$$\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta).$$

3.
$$\tan(\alpha \pm \beta) = \frac{\tan(\alpha) \pm \tan(\beta)}{1 \mp \tan(\alpha)\tan(y)}.$$

**Double angle**

1. $$\sin(2\theta) = 2\sin(\theta)\cos(\theta);$$

2. $$\begin{aligned}\cos(2\theta) &= \cos^2(\theta) - \sin^2(\theta)\\ &= 2\cos^2(\theta) - 1\\ &= 1 - 2\sin^2(\theta).\end{aligned}$$

### 2.8.4 Inverse trigonometric functions

**Definition 2.16.** The trigonometric functions are not bijective so, they are not invertible on their full domains. By restricting the domain the inverse trigonometric functions are defined as follows:

- $$\arcsin(x) : [-1, 1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right];$$

- $$\arccos(x) : [-1, 1] \to [0, \pi];$$

- $$\arctan(x) : \mathbb{R} \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

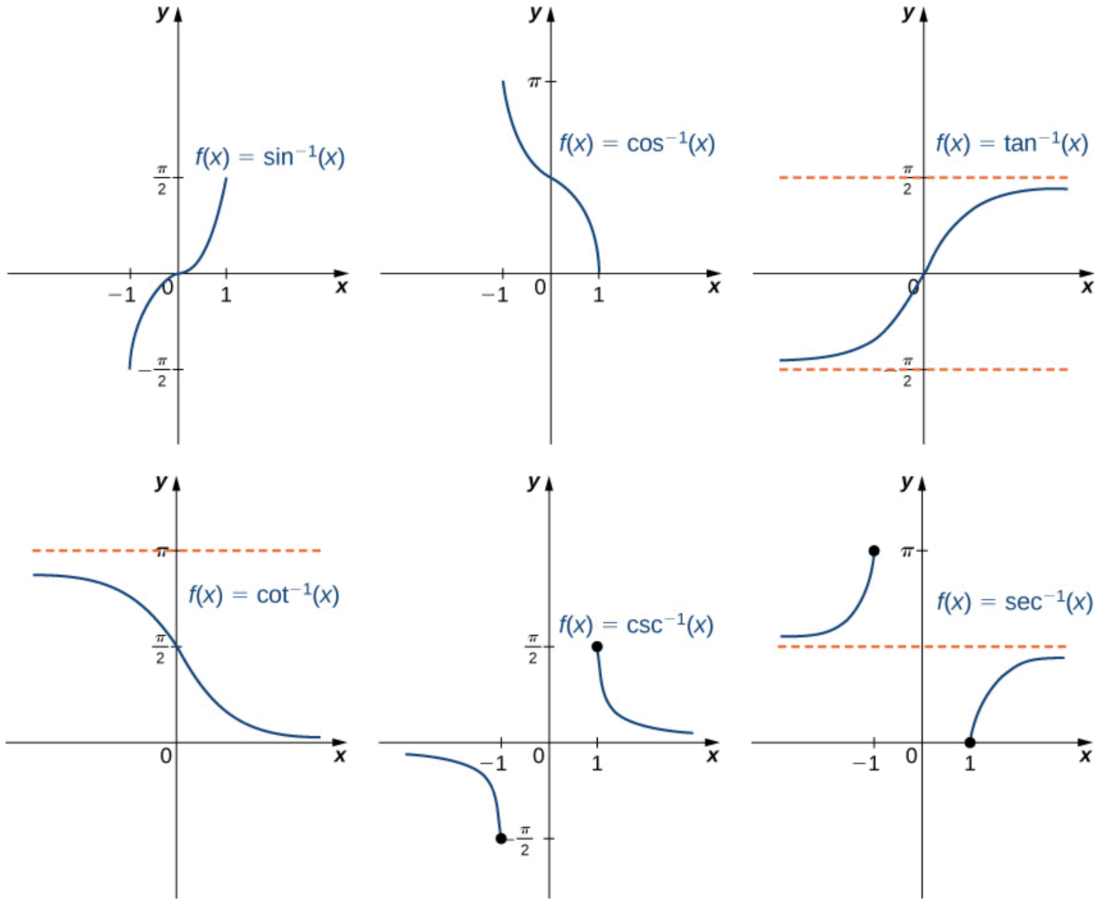


Figure 5: The graph of each of the inverse trigonometric functions

### 2.8.5 Trigonometric functions with Complex variables

**Definition 2.17. Euler's formula** is

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

where $\theta \in \mathbb{R}$.

From Euler's formula it is possible to derive the following identities:

- $$\sin(x) = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right);$$
- $$\cos(x) = \frac{1}{2}\left(e^{ix} + e^{-ix}\right).$$

## 2.9 Hyperbolic functions

**Definition 2.18.** The hyperbolic sine and cosine are defined as follows:

$$\begin{aligned} \sinh(x) : \mathbb{R} \to \mathbb{R} \quad \sinh(x) &= \frac{e^x - e^{-x}}{2} \\ \cosh(x) : \mathbb{R} \to [1, \infty) \quad \cosh(x) &= \frac{e^x + e^{-x}}{2}. \end{aligned}$$

**Definition 2.19.** The hyperbolic tangent is defined as:

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

where $\tanh : \mathbb{R} \to (-1, 1)$.

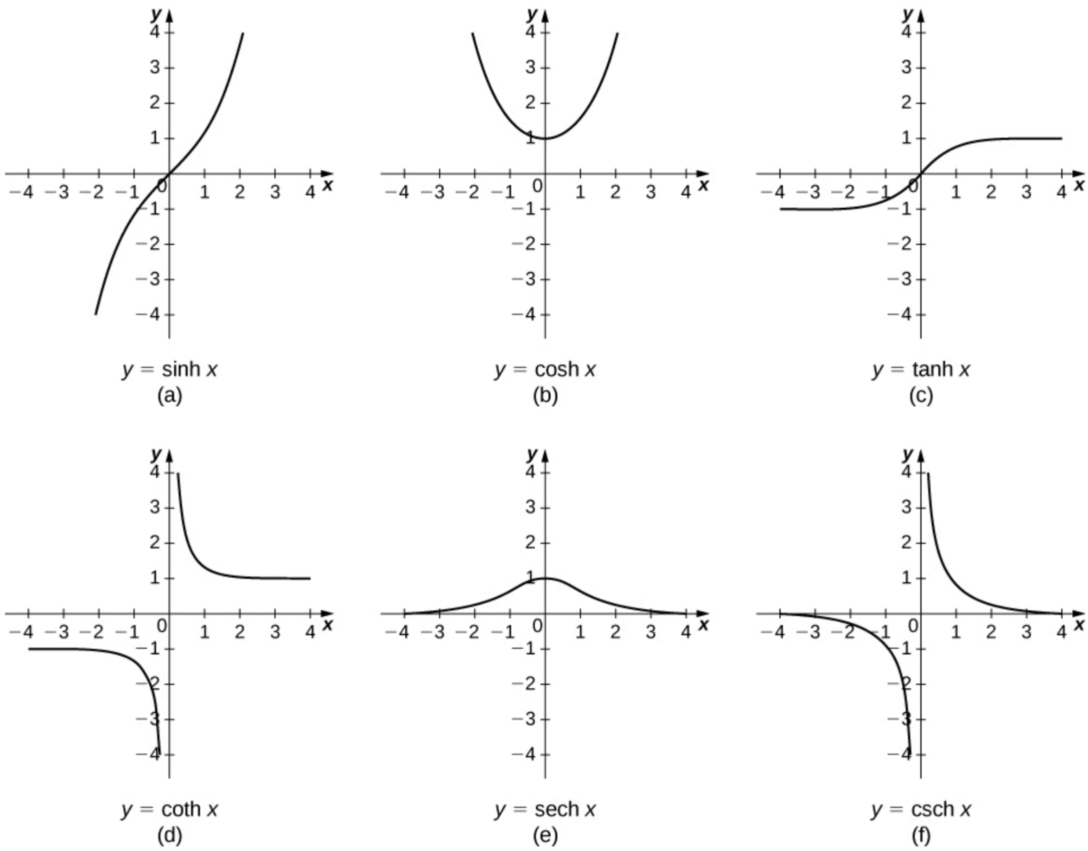


Figure 6: Graphs of hyperbolic functions

### 2.9.1 Inverse hyperbolic functions

**Definition 2.20.** The inverse hyperbolic sine, cosine and tangent are defined as follows:

$$\begin{aligned}
\operatorname{arcsinh}(x) &= \ln\left(x + \sqrt{1+x^2}\right)\\
\operatorname{arccosh}(x) &= \ln\left(x + \sqrt{x^2-1}\right)\\
\operatorname{arctanh}(x) &= \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right).
\end{aligned}$$

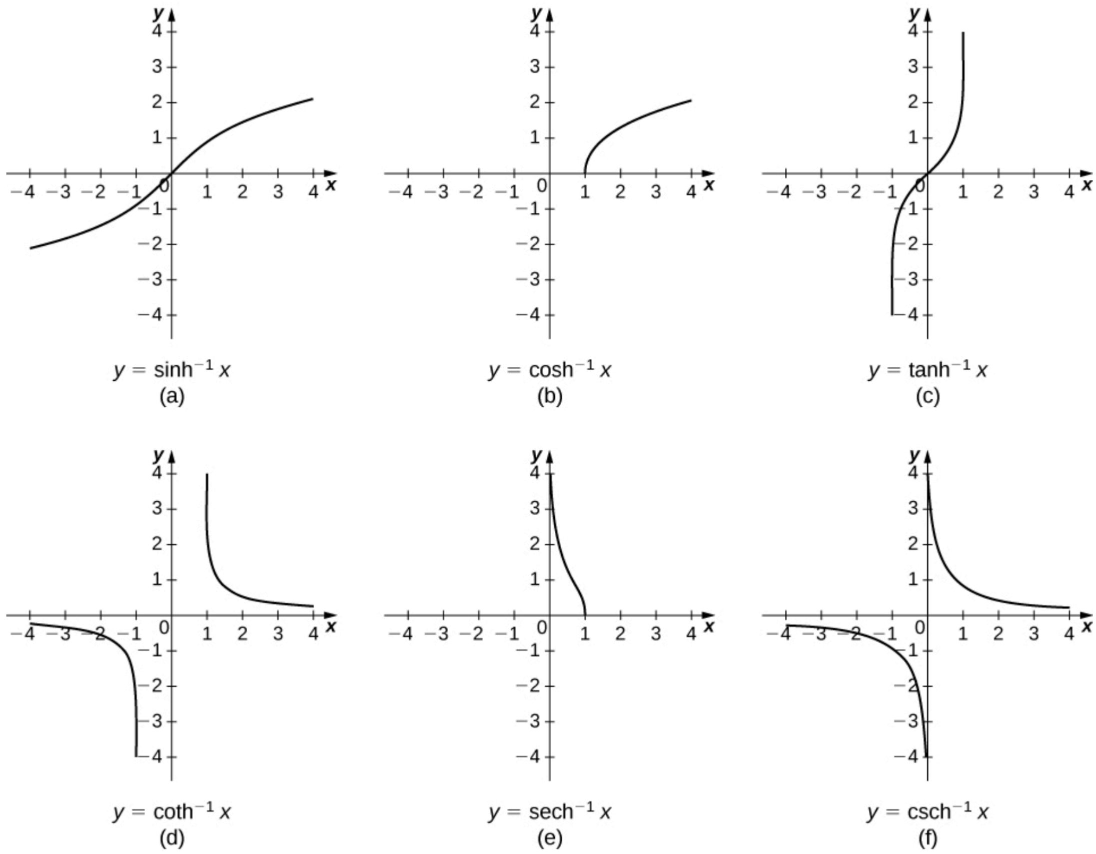


Figure 7: Graphs of the inverse hyperbolic functions

**Example.** Find the inverse of the hyperbolic sine function.
**Solution:** Begin by writing $y = \frac{1}{2}(e^x - e^{-x})$ and rearrange the expression to find

$x$ as a function of $y$.

$$\begin{aligned} e^x - e^{-x} - 2y &= 0 \\ e^{2x} - 1 - 2ye^x &= 0 \\ (e^x - y)^2 - y^2 - 1 &= 0 \quad \text{(by completing the square)} \\ \Rightarrow e^x - y &= \sqrt{1+y^2} \\ x &= \ln\Big(y + \sqrt{1+y^2}\Big). \end{aligned}$$

### 2.9.2 Hyperbolic identities

1. $$\cosh^2(x) - \sinh^2(x) = 1$$

2. $$1 - \tanh^2(x) = \operatorname{sech}^2(x)$$

3. $$\coth^2(x) - 1 = \operatorname{cosech}^2(x)$$

#### Addition and subtraction

1. $$\sinh(x \pm y) = \sinh(x)\cosh(y) \pm \cosh(x)\sinh(y)$$

2. $$\cosh(x \pm y) = \cosh(x)\cosh(y) \pm \sinh(x)\sinh(y)$$

3. $$\tanh(x \pm y) = \frac{\tanh(x) \pm \tanh(y)}{1 \pm \tanh(x)\tanh(y)}$$

#### Double ‘angle’

1. $$\sinh(2x) = 2\sinh(x)\cosh(x)$$

2. $$\begin{aligned} \cosh(2x) &= \cosh^2(x) + \sinh^2(x) \\ &= 2\cosh^2(x) - 1 \\ &= 2\cosh^2(x) - 1 \end{aligned}$$

### 2.9.3 Hyperbolic functions with Complex variables

Using Euler's formula the following identities can be derived:

1. $$\sin x = -i \sinh(ix)$$

2. $$\cos x = \cosh(ix)$$

3. $$\tan x = -i \tanh(ix)$$

4. $$\sinh x = -i \sin(ix)$$

5. $$\cosh x = \cos(ix)$$

6. $$\tanh x = -i \tan(ix)$$

# 3 Limits

**Definition 3.1.** The **right limit**:

$$\lim_{x \to a^+} = L \iff \forall \varepsilon > 0 \ \exists \delta > 0 \text{ such that } 0 < x - a < \delta \Rightarrow |f(x) - L| < \varepsilon.$$

**Definition 3.2.** The **left limit**:

$$\lim_{x \to a^-} = L \iff \forall \varepsilon > 0 \ \exists \delta > 0 \text{ such that } 0 < a - x < \delta \Rightarrow |f(x) - L| < \varepsilon.$$

**Definition 3.3.** The **limit**:

$$\lim_{x \to a} f(x) = L \iff \forall \varepsilon > 0 \ \exists \delta > 0 \text{ such that } 0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon.$$

**Remark.** *The limit of a function exists if and only if both the left and right limit exist and they are equal.*

**Example.** Prove $\lim_{x\to 1}(3x-1) = 2$ with the definition of the limit.
**Solution:** Proceed by choosing a suitable $\delta$ for which the definition of the limit for $f(x) = 3x - 1$ is satisfied.
*Rough work:*

$$\begin{aligned} |f(x) - L| < \varepsilon &\Rightarrow |(3x-1) - 2| < \varepsilon \\ &\Rightarrow |3x - 3| < \varepsilon \\ &\Rightarrow 3|x-1| < \varepsilon \\ &\Rightarrow |x-1| < \frac{\varepsilon}{3}. \end{aligned}$$

In this case, choose $\delta = \frac{\varepsilon}{3}$ then,

$$\begin{aligned} |x-1| < \delta \Rightarrow |x-1| &< \frac{\varepsilon}{3} \\ &\Rightarrow 3|x-1| < \varepsilon \\ &\Rightarrow |3x-3| < \varepsilon \\ &\Rightarrow |(3x-1)-2| < \varepsilon. \end{aligned}$$

The proof is complete.

## 3.1 Limits involving infinity

**Definition 3.4.**

$$\lim_{x\to\infty} f(x) = L \iff \forall\varepsilon > 0,\ \forall x > X,\ \exists X > 0 \text{ such that } |f(x) - L| < \varepsilon.$$

**Definition 3.5.**

$$\lim_{x\to-\infty} f(x) = L \iff \forall\varepsilon > 0,\ \forall x < X,\ \exists X < 0 \text{ such that } |f(x) - L| < \varepsilon.$$

## 3.2 Algebra of limits

**Theorem 3.1.** For any real number $a$ and any constant $c$,

1. $$\lim_{x\to a} x = a;$$

2. $$\lim_{x\to a} c = c.$$

**Theorem 3.2.** Let $f(x)$ and $g(x)$ be defined for all $x \neq a$ over some open interval containing $a$. Assume that $L$ and $M$ are real numbers such that $lim_{x\to a} f(x) = L$ and $lim_{x\to a} g(x) = M$. Let $c$ be a constant. Then, each of the following statements holds:

1. Sum and Difference law:
$$\lim_{x\to a}(f(x) \pm g(x)) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) = L \pm M;$$
2. Constant multiple law:
$$\lim_{x\to a} cf(x) = c \cdot \lim_{x\to a} f(x) = cL;$$
3. Product law:
$$\lim_{x\to a}(f(x)g(x)) = \left(\lim_{x\to a} f(x)\right)\left(\lim_{x\to a} g(x)\right) = LM;$$
4. Quotient law:
$$\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right)\frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{L}{M}$$
for $M \neq 0$;
5. Power law:
$$\lim_{x\to a}(f(x))^n = \left(\lim_{x\to a} f(x)\right)^n = L^n$$
for every positive integer $n$;
6. Root law:
$$\lim_{x\to a}\sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)} = \sqrt[n]{L}$$
for all $L$ if $n$ is odd and for $L \geq 0$ if $n$ is even.

### 3.2.1 Limits of composite functions

**Theorem 3.3.** If $f(x)$ is continuous at $L$ and $\lim_{x\to a} g(x) = L$, then
$$\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right) = f(L).$$

## 3.3 Multiple limits

**Theorem 3.4.** For functions of multiple variables the limits are defined as:
$$\lim_{x\to a}\lim_{y\to b} f(x,y) := \lim_{x\to a}\left(\lim_{y\to b} f(x,y)\right).$$

**Remark.** *In general*
$$\lim_{x\to a}\lim_{y\to b} f(x,y) \neq \lim_{y\to b}\lim_{x\to a} f(x,y).$$

**Example.** Show that changing the order the limits are taken in the following double limit changes its value:

$$\lim_{x\to\infty}\lim_{y\to-\infty}(1+\tanh(x+y)).$$

**Solution:**

$$\begin{aligned}\lim_{x\to\infty}\left(\lim_{y\to-\infty}(1+\tanh(x+y))\right) &= \lim_{x\to\infty}(1-1)=0 \quad \text{whereas,}\\ \lim_{y\to-\infty}\left(\lim_{x\to\infty}(1+\tanh(x+y))\right) &= \lim_{y\to-\infty}(1+1)=2.\end{aligned}$$

## 3.4 Continuous functions

**Definition 3.6.** A function $f(x)$ is **continuous at the point** $x=a$ if $\lim_{x\to a}f(x)$ exists and is equal to $f(a)$.

**Definition 3.7.** A function $f(x)$ is **continuous** if it is continuous at all points in its domain.

## 3.5 Intermediate Value Theorem

**Theorem 3.5.** Let $f(x)$ be a continuous function on $[a,b]$ and suppose $f(a)<y<f(b)$ then there exists an $x=c$ with $c\in(a,b)$ such that $f(c)=y$.

**Example.** Show that there is a root of the equation $4x^3-6x^2+3x-2=0$ in the interval $[1,2]$.
**Solution:** Consider the function $f(x)=4x^3-6x^2+3x-2$over the closed interval $[1,2]$. The function $f$ is a polynomial, therefore it is continuous over $[1,2]$. It follows that $f(1)=-1$ and $f(2)=12$. Since, $f(1)<0<f(2)$ by the Intermediate Value Theorem there exists a value $c$ in the interval $(1,2)$ such that $f(c)=0$, i.e. there is a solution for the equation $f(x)=0$ in the interval $(1,2)$.

## 3.6 Sandwich Theorem

**Theorem 3.6.** Suppose

$$f(x)\le g(x)\le h(x)$$

for all $x$ such that $0<|x-a|<\delta$ with some $\delta>0$, and

$$\lim_{x\to a}f(x)=\lim_{x\to a}h(x)=L,$$

where $L\in\mathbb{R}$. Then

$$\lim_{x\to a}g(x)=L.$$

**Example.** Find
$$\lim_{x\to 0} x^2 \cos\left(\frac{1}{x^2}\right).$$
**Solution:** When trying to find functions to use to 'sandwich', the functions must be similar enough to $g(x)$ and easier to evaluate their limit as $x \to a$. To do this, start with the most complicated part of $g(x)$, to find constants or simpler functions that it stays between.
In this case the cosine function is bounded by $-1$ and 1, so
$$-1 \le \cos\left(\frac{1}{x^2}\right) \le 1$$
$$-x^2 \le x^2\cos\left(\frac{1}{x^2}\right) \le x^2.$$
Since
$$\lim_{x\to 0} -x^2 = \lim_{x\to 0} x^2 = 0$$
by the Sandwich theorem,
$$\lim_{x\to 0} x^2 \cos\left(\frac{1}{x^2}\right)$$

## 3.7 Limits to learn

### 3.7.1 A common trick

A common trick to evaluate limits:
$$\lim_{x\to a} f(x)^{g(x)} = \lim_{x\to a} e^{\ln f(x)^{g(x)}} \lim_{x\to a} e^{g(x)\ln f(x)} = e^{\lim_{x\to a}(g(x)\ln f(x))}.$$

### 3.7.2 Standard limits

1. $$\lim_{x\to 0}\frac{\sin(x)}{x} = 1$$
2. $$\lim_{x\to 0}\frac{e^x - 1}{x} = 1$$
3. $$\lim_{x\to 0}\frac{\cos(x) - 1}{x} = 0$$

### 3.7.3 Common Limits

1.
$$\lim_{x\to 0} \frac{\tan(x)}{x} = 1$$

2.
$$\lim_{x\to 0} \frac{\ln(1+x)}{x} = 0$$

3.
$$\lim_{x\to \infty} \left(1+\frac{1}{x}\right)^x = e$$

# 4 Differentiation

## 4.1 Differentiation from First Principles

**Definition 4.1.** The **derivative** of a function $f(x)$ with respect to $x$ is

$$\frac{\mathrm{d}f}{\mathrm{d}x} \equiv \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}.$$

## 4.2 Differentiable functions

**Definition 4.2.** A function $f(x)$ is **differentiable** at the point $x = a$ if the derivative $\frac{\mathrm{d}f}{\mathrm{d}x}$ exists at $x = a$. A function for which the derivative exists for all points in its domain is called a **differentiable function**.

**Theorem 4.1.** If a function $f(a,b) \to \mathbb{R}$ is differentiable on the interval $(a,b)$, then it is continuous on $(a,b)$.

## 4.3 Properties of derivatives

**Theorem 4.2.** Let $f$ and $g$ be differentiable functions.

- The product rule: $(fg)' = gf' + fg'$;
- the quotient rule: $\left(\frac{f}{g}\right) = \left(\frac{gf'-fg'}{g^2}\right)$ if $g \neq 0$.

## 4.4 Derivatives inverse functions

**Theorem 4.3.** Let $f(x)$ be a function that is both invertible and differentiable. Let $y = f^{-1}(x)$ be the inverse of $f(x)$. For all $x$ satisfying $f'\left(f^{-1}(x)\right) \neq 0$,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left(f^{-1}(x)\right) = \frac{1}{f'\left(f^{-1}(x)\right)}.$$

Alternatively, if $y = g'(x)$ is the inverse of $f(x)$, then

$$g(x) = \frac{1}{f'(g(x))}.$$

**Example.** Find the derivative of $g(x) = \arcsin(x)$.
**Solution:** Since for $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $f(x) = \sin(x)$ is the inverse of $g(x) = \arcsin(x)$, begin by finding $f'(x)$.

$$\begin{aligned} f(x) &= \sin(x) \\ f'(x) &= \cos(x) \\ f'(g(x)) &= \cos(\arcsin(x)) = \sqrt{1-x^2}. \end{aligned}$$

Therefore,

$$g'(x) = \frac{\mathrm{d}}{\mathrm{d}x}(\arcsin(x)) = \frac{1}{f'(g(x))} = \frac{1}{\sqrt{1-x^2}}.$$

## 4.5 Mean Value Theorem

**Theorem 4.4.** Let $f$ be a continuous function on $[a, b]$ and a differentiable function on $(a, b)$, then there exists a point $c \in (a, b)$ where

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

**Corollary 4.4.1.** *Let $f$ be differentiable over an interval $I$. If $f'(x) = 0$ for all $x \in I$, then $f(x)$ is constant for all $x \in I$.*

### 4.5.1 Rolle's Theorem

**Theorem 4.5.** Let $f$ be a continuous function over the closed interval $[a, b]$ and differentiable over the open interval $(a, b)$ such that $f(a) = f(b)$.There then exists at least one $c \in (a, b)$ such that $f'(c) = 0$.

**Remark.** *This is a special case of the mean value theorem.*

**Example.** Show that $f(x) = x^3 - 7x^2 + 25x + 8$ has exactly one real root.
**Solution:** Note that $f(0) = 8$ and $f(-1) = -25$ so, $f(-1) < 0 < f(0)$. Therefore, by the Intermediate Value Theorem there exists a $c \in (-1, 0)$ such that $f(c) = 0$. This proves that the function has at least one real root. Assume there is more than one real root hence, there exists $a$ and $b$, such that

$$f(a) = f(b) = 0.$$

Since, $f(x)$ s a polynomial it is continuous and differentiable everywhere by Rolle's theorem there exists a $c$ such that $f'(c) = 0$. If such $c$ exists then

$$\begin{aligned} f'(x) &= 3x^2 - 14x + 25 \\ \Rightarrow f'(c) &= 3c^2 - 14c + 25 = 0 \\ \Rightarrow c &= \frac{7 \pm \sqrt{26}i}{3}. \end{aligned}$$

No such $c \in \mathbb{R}$ exists so, $f(x)$ has exactly one real root.

## 4.6 Standard derivatives

### 4.6.1 Trigonometric derivatives

1. $$\frac{\mathrm{d}(\arcsin x)}{\mathrm{d}x} = \frac{1}{\sqrt{1 - x^2}}$$

2. $$\frac{\mathrm{d}(\arccos x)}{\mathrm{d}x} = -\frac{1}{\sqrt{1 - x^2}}$$

3. $$\frac{\mathrm{d}(\arctan x)}{\mathrm{d}x} = \frac{1}{\sqrt{1 + x^2}}$$

4. $$\frac{\mathrm{d}(\sec x)}{\mathrm{d}x} = \sec x \tan x$$

5. $$\frac{\mathrm{d}(\cot x)}{\mathrm{d}x} = -\operatorname{cosec}^2 x$$

6. $$\frac{\mathrm{d}(\operatorname{cosec} x)}{\mathrm{d}x} = -\operatorname{cosec} x \cot x$$

#### 4.6.2 Hyperbolic derivatives

1. $$\frac{\mathrm{d}(\tanh(x))}{\mathrm{d}x} = \operatorname{sech}^2(x)$$

2. $$\frac{\mathrm{d}(\coth(x))}{\mathrm{d}x} = -\operatorname{cosech}^2(x)$$

3. $$\frac{\mathrm{d}(\operatorname{sech}(x))}{\mathrm{d}x} = -\operatorname{sech}(x)\tanh(x)$$

4. $$\frac{\mathrm{d}(\operatorname{cosech}(x))}{\mathrm{d}x} = -\operatorname{cosech}(x)\coth(x)$$

5. $$\frac{\mathrm{d}(\operatorname{arcsinh} x)}{\mathrm{d}x} = \frac{1}{\sqrt{1+x^2}}$$

6. $$\frac{\mathrm{d}(\operatorname{arccosh} x)}{\mathrm{d}x} = \frac{1}{\sqrt{x^2-1}}$$

7. $$\frac{\mathrm{d}(\operatorname{arctanh} x)}{\mathrm{d}x} = \frac{1}{1-x^2}$$

# 5 Integration

**Definition 5.1.** The **integral** $\int_a^b f(x)\,dx$ is the total **signed** area between the graph $y = f(x)$ and the $x$-axis in the $(x, y)$ plane between $x = a$ and $x = b$, counted positively for $f(x) > 0$ and negatively for $f(x) < 0$.

**Theorem 5.1.** If $f : [a, b] \to \mathbb{R}$ is continuous, then $\int_a^b f(x)\,dx$ exists.

## 5.1 Riemann Integral

## 5.2 Fundamental theorem of Calculus

**Theorem 5.2. FTC I** Let $f : [a, b] \to \mathbb{R}$ be a continuous function and let $F : [a, b] \to \mathbb{R}$ be defined by

$$F(x) = \int_a^x f(t)\,dt.$$

Then, F is a continuous function on $[a, b]$, differentiable on the interval $(a, b)$ and

$$\frac{dF}{dx} = f(x).$$

**Remark.** *The function $F(x)$ is called the **antiderivative** or the **primitive** of $f(x)$.*

**Example.** Let

$$F(x) = \int_0^{x^3} \sin\left(e^t\right) dt.$$

Find $F'(x)$.
**Solution:** Let

$$H(x) = \int_0^{x} \sin\left(e^t\right) dt$$

then, by FTC I

$$H'(x) = \sin(e^x).$$

Note that

$$\begin{aligned} F(x) &= H(x^3) \\ \Rightarrow F'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}(H(x^3)) \\ &= 3x^2 H'(x^3) \quad \text{(by the chain rule)} \\ &= 3x^2 \sin\left(e^{x^3}\right). \end{aligned}$$

**Example.** Let

$$F(x) = \int_0^{\cosh(x)} e^{\sqrt{t}+x^2}\, dt.$$

Write $F'(x)$ in terms of $F(x)$.
**Solution:** Notice that

$$\begin{aligned} F(x) &= \int_0^{\cosh(x)} e^{\sqrt{t}+x^2}\, dt \\ &= e^{x^2} \int_0^{\cosh(x)} e^{\sqrt{t}}\, dt. \end{aligned}$$

Let

$$H(x) = \int_0^{x} e^{\sqrt{t}\, dt.}$$

Now

$$\begin{aligned} F(x) &= e^{x^2} H(\cosh(x)) \\ F'(x) &= 2xe^{x^2} H(\cosh(x)) + e^{x^2} H'(\cosh(x)) \sinh(x) \\ &= 2xF(x) + e^{x^2} e^{\sqrt{x}} \sinh(x). \end{aligned}$$

Therefore,

$$F'(x) = 2xF(x) + e^{x^2+\sqrt{x}} \sinh(x).$$

**Theorem 5.3.** Let $f : [a, b] \to \mathbb{R}$ be a continuous function and let $F : [a, b] \to \mathbb{R}$ be defined by

$$\frac{dF}{dx} = f(x) \quad \forall x \in [a, b].$$

Then, if $f(x)$ is integrable on $[a, b]$,

$$\int_a^b f(x)\, dx = F(b) - F(a).$$

## 5.3 Integration techniques

### 5.3.1 t-substitution

For complicated integrals involving trigonometric functions we can use the following substitution.
Let $t = \tan \frac{x}{2}$, then:

$$\sin x = \frac{2t}{1+t^2}; \quad \cos x = \frac{1-t^2}{1+t^2}; \quad \frac{dx}{dt} = \frac{2}{1+t^2}$$

### 5.3.2 Integration by parts

**Theorem 5.4.** Let $u = f(x)$ and $v = g(x)$ be functions with continuous derivatives. Then, the integration-by-parts formula for the integral involving these two functions is:

$$\int u \frac{\mathrm{d}v}{\mathrm{d}x}\, dx = uv - \int v \frac{\mathrm{d}u}{\mathrm{d}x} dx$$

**Remark.** *The acronym **LIATE** can be used to choose which function to differentiate. (The I stands inverse trig/hyp functions).*

### 5.3.3 Partial fractions

| Factor in denominator | Term in partial fraction decomposition |
| --- | --- |
| $ax+b$ | $\frac{A}{ax+b}$ |
| $(ax+b)^k$ | $\frac{A_1}{ax+b}+\frac{A_2}{(ax+b)^2}+\cdots+\frac{A_k}{(ax+b)^k}, k=1,2,3,\ldots$ |
| $ax^2+bx+c$ | $\frac{Ax+B}{ax^2+bx+c}$ |
| $\left(ax^2+bx+c\right)^k$ | $\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots+\frac{A_kx+B_k}{(ax^2+bx+c)^k}, k=1,2,3,\ldots$ |

Figure 8: Table of partial fraction decomposition.

## 5.4 Volume of revolution

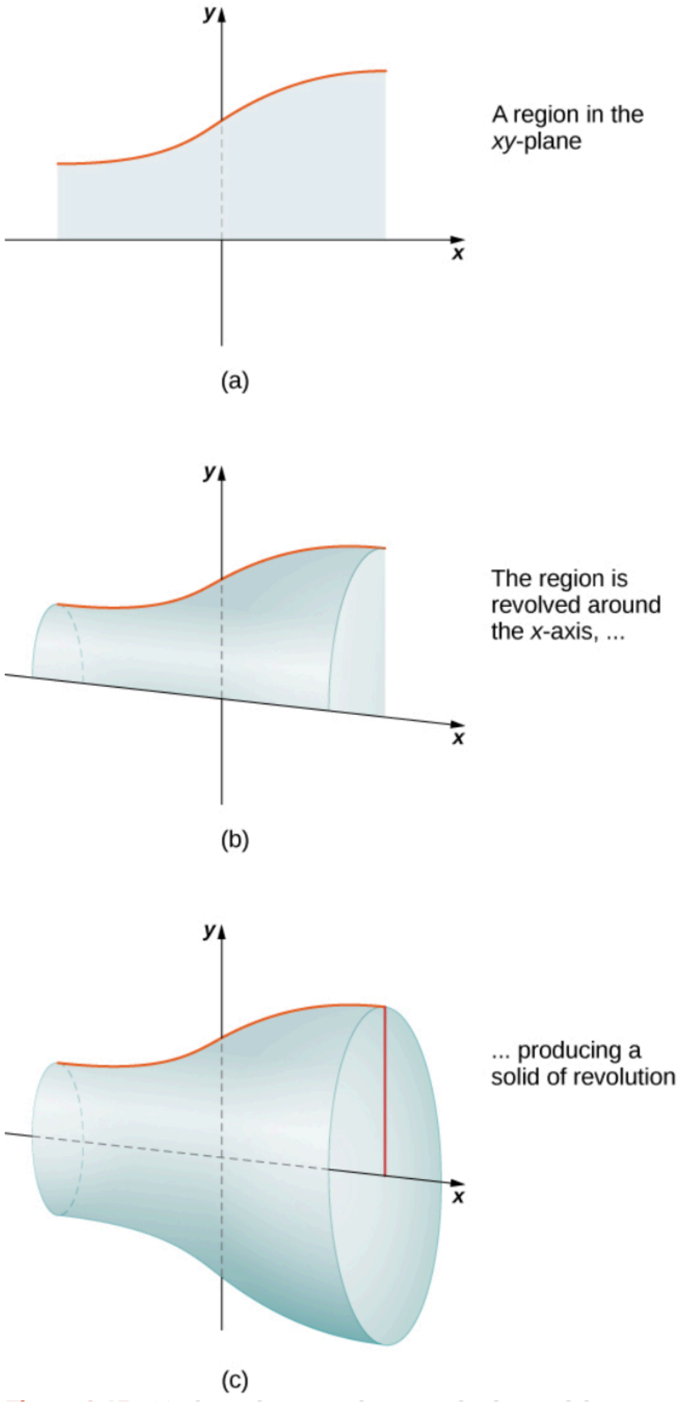


Figure 9: Illustration of a graph's revolution

**Theorem 5.5.** Let $f(x)$ be continuous and non negative. Define $R$ as the region bounded above by the graph of $f(x)$, below by the $x$-axis, on the left by the line $x = a$, and on the right by the line $x = b$. Then, the volume of the solid of revolution formed by revolving $R$ around the $x$-axis is given by

$$V = \int_a^b \pi \left[f(x)\right]^2.$$

**Remark.** *For the volume of revolution around the $y$-axis, $V$ is evaluated by*

$$V = \int_c^d \pi \left[g(y)\right]^2 \, dy.$$

## 5.5 Length of a curve

### 5.5.1 Cartesian coordinates

**Theorem 5.6.** Let $f(x)$ be a smooth function over the interval $[a, b]$. Then the **arc length** of the portion of the graph of $f(x)$ from the point $(a, f(a))$ to the point $(b, f(b))$ is given by

$$s = \int_a^b \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} \, dx.$$

### 5.5.2 Parametric coordinates

**Theorem 5.7.**

$$s = \int_a^b \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2} \, dt.$$

### 5.5.3 Polar coordinates

**Theorem 5.8.**

$$s = \int_a^b \sqrt{r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2} \, d\theta.$$

# 6 Power Series

## 6.1 Infinite sums

**Definition 6.1.** An **infinite series** is an expression of the form $\sum_{n=n_0}^{\infty} a_n$ where $n, n_0 \in \mathbb{Z}$. For a real series $a_n \in \mathbb{R}$.

**Definition 6.2.** A **partial sum** of an infinite series $S \equiv \sum_{n=n_0}^{\infty} a_n$ has the form $S_N \equiv \sum_{n=n_0}^{N} a_n$.

## 6.2 Convergence

**Definition 6.3.** A series $S \equiv \sum_{n=n_0}^{\infty} a_n$ is called **convergent** if the limit $S \equiv \lim_{N\to\infty}(S_N)$ exists, where$S_N \equiv \sum_{n=n_0}^{N} a_n$. If a series is not convergent it is called **divergent**.

**Lemma 6.1.** *If an infinite series $S \equiv \sum_{n=n_0}^{\infty} a_n$ is convergent then,*

$$\lim_{n\to\infty}(a_n) = 0.$$

*Proof.* Consider,

$$\lim_{n\to\infty}(S_n - S_{n-1}) = \lim_{n\to\infty}(S_n) - \lim_{n\to\infty}(S_{n-1}) = S - S = 0.$$

□

**Definition 6.4.** A series $\sum_{n=n_0}^{\infty} a_n$ is **absolutely convergent** if the series $\sum_{n=n_0}^{\infty} |a_n|$ converges.

**Theorem 6.2.** Every absolutely convergent series is convergent, i.e. if $\sum_{n=n_0}^{\infty} |a_n|$ converges then $\sum_{n=n_0}^{\infty} a_n$ is also convergent.

**Theorem 6.3.** Let $\sum_{n=n_0}^{\infty} b_n$ be a convergent series of non-negative numbers. Suppose there exists $M >)$ and $N \in \mathbb{Z}$ such that

$$|a_n| \leq M b_n, \quad \forall n \geq N.$$

Then $\sum_{n=n_0}^{\infty} a_n$ is absolutely convergent. *(Theorem 13.4 of Sequences and series lecture notes).*

**Theorem 6.4.** Suppose $b_n \geq 0$ for all $n \geq n_0$ and there exists $C > 0$ such that

$$\sum_{n=n_0}^{N} b_n \leq C \quad \forall N \geq n_0.$$

Then the series $\sum_{n=n_0}^{N} b_n$ converges and its sum is $\leq C$.

## 6.3 Series convergence criteria

### 6.3.1 Limit comparison test

**Theorem 6.5.** Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be positive-term series ($a_n > 0$ and $b_n > 0$) such that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L \neq 0.$$

Then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} b_n$ converges.

### 6.3.2 The n'th Root Test or Cauchy's Criterion

**Theorem 6.6.** Let $\sum_{n=1}^{\infty} a_n$ be a positive-term series. Then

$$\lim_{n\to\infty} |a_n|^{\frac{1}{n}} \begin{cases} < 1 \sum_{n=1}^{\infty} a_n & \text{is convergent,} \\ = 1 & \text{test is inconclusive,} \\ > 1 \sum_{n=1}^{\infty} a_n & \text{is divergent.} \end{cases}$$

### 6.3.3 The Ratio Test or D'Alembert's Criterion

**Theorem 6.7.** Let $\sum_{n=1}^{\infty} a_n$ be a series of positive-terms and for each $n \in \mathbb{N}$ let $\alpha_n = \frac{a_{n+1}}{a_n}$.

$$\lim_{n\to\infty} |\alpha_n| \begin{cases} < 1 \sum_{n=1}^{\infty} a_n & \text{is convergent,} \\ = 1 & \text{test is inconclusive,} \\ > 1 \sum_{n=1}^{\infty} a_n & \text{is divergent.} \end{cases}$$

## 6.4 Series as a function of $x$

**Definition 6.5.** A function which can be locally written as a convergent power series is called an **analytic function**.

**Definition 6.6.** A **power series** is an expression of the form $S(x) \equiv \sum_{n=n_0}^{\infty} b_n x^n$.

**Theorem 6.8.** For every power series $\sum_{n=n_0}^{\infty} b_n x_n$ there exists $R \geq 0$ or $R = +\infty$ such that the series is absolutely convergent for all $x$ with $|x| < R$ and is divergent for all $x$ with $|x| > R$.

**Definition 6.7.** The $R$ from the previous theorem is known as **radius of convergence** of a power series $\sum_n b_n x^n$ where

$$R = \lim_{n\to\infty} |b_n|^{-\frac{1}{n}}$$

and, where it exists,

$$R = \lim_{n\to\infty} \left| \frac{b_n}{b_{n+1}} \right|.$$

**Remark.** *What happens when*

$$R = \begin{cases} |x| & \textit{inconclusive} \\ 0 & \textit{the series diverges for all } x \neq 0 \\ \infty & \textit{the series converges for all } x \in \mathbb{R}. \end{cases}$$

**Example.** Find the radius of convergence for

$$S(x) = \sum_{n=0}^{\infty} \frac{x^n}{n}.$$

**Solution:** Here $b_n = \frac{1}{n}$ so,

$$\begin{aligned} R &= \lim_{n\to\infty} \left| \frac{b_n}{b_{n+1}} \right| \\ &= \lim_{n\to\infty} \left| \frac{\frac{1}{n}}{\frac{1}{n+1}} \right| \\ &= \lim_{n\to\infty} \frac{n+1}{n} \\ &= \lim_{n\to\infty} \left( 1 + \frac{1}{n} \right) \\ &= 1. \end{aligned}$$

Hence, the series converges absolutely for $|x| < 1$ and diverges for $|x| > 1$. At $x = 1$ the series $S(x) = \sum_{n=0}^{\infty} \frac{1}{n}$, so the series is divergent.

## 6.5 Taylor's theorem

**Definition 6.8.** If $f$ has derivatives up to order $N$ at $x = a$, then the **Taylor series** to order $N$ for the function $f$ at $a$ is

$$f(x) = \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n + R_{N+1}(x) \quad \text{where } R_{N+1} = \frac{1}{N!} \int_a^x f^{N+1}(y)(x-y)^N \, dy.$$

### 6.5.1 Maclaurin series

**Definition 6.9.** The **Maclaurin series** for a function $f(x)$ is a Taylor series expansion about $x = 0$ and to order $N$ is written

$$f(x) = \sum_{n=0}^{N} \frac{f^{(n)}(0)}{n!} x^n + R_{N+1}(x) \quad \text{where } R_{N+1} = \frac{1}{N!} \int_a^x f^{N+1}(y)(x-y)^N \, dy.$$

### 6.5.2 Common Maclaurin's series

1.
$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \quad x \in \mathbb{R}$$

2.
$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1!)} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \dots \quad x \in \mathbb{R}$$

3.
$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n!)} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \quad x \in \mathbb{R}$$

4.
$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n!} = x - \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!} \dots \quad x \in (-1, 1]$$

5.
$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots \quad x \in (-1, 1)$$

# 7 L'Hôpital's Rule

**Theorem 7.1.** Suppose $f$ and $g$ are differentiable functions over an open interval containing $a$, except possibly at $a$. If $\lim_{x\to a} f(x) = 0$ and $l \lim_{x\to a} g(x) = 0$, then

$$\lim_{x\to a} \left( \frac{f(x)}{g(x)} \right) = \lim_{x\to a} \left( \frac{f'(x)}{g'(x)} \right),$$

assuming the limit on the right exists or is $\pm\infty$.

**Theorem 7.2.** Suppose $f$ and $g$ are differentiable functions over an open interval containing $a$, except possibly at $a$. If $\lim_{x\to a} f(x) = \pm\infty$ and $l \lim_{x\to a} g(x) = \pm\infty$, then

$$\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right) = \lim_{x\to a}\left(\frac{f'(x)}{g'(x)}\right),$$

assuming the limit on the right exists or is $\pm\infty$.